

Indefinite symmetric spaces with $G_{2(2)}$ -structure

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Abstract

We determine all indecomposable pseudo-Riemannian symmetric spaces of signature $(4, 3)$ whose holonomy is contained in $G_{2(2)} \subset SO(4, 3)$.

1 Introduction

The compact Lie group G_2 lies on the list of holonomy groups of irreducible Riemannian manifolds. Each Riemannian manifold whose holonomy group is contained in G_2 is Ricci-flat. In particular, if the holonomy group of a Riemannian symmetric space M is contained in G_2 , then M must be flat. In contrast, we will see that there exist indecomposable indefinite symmetric spaces of signature $(4, 3)$ whose holonomy is contained in the split real form $G_{2(2)} \subset SO(4, 3)$. The same effect appears also with other holonomy groups known from Riemannian geometry, e.g, there are Ricci-flat hermitian symmetric spaces (see [KO2] for index 2) and hyper-Kähler symmetric spaces, see [AC1, KO3]. Of course, in all cases the holonomy group of the symmetric space is properly contained in $G_{2(2)}$, $SU(p, q)$, and $Sp(p, q)$, respectively.

Let us recall the definition of the group $G_{2(2)}$. We consider the generic 3-form

$$\omega_0 = \sqrt{2}(\sigma^{127} + \sigma^{356}) - \sigma^4 \wedge (\sigma^{15} + \sigma^{26} - \sigma^{37}). \quad (1)$$

on \mathbb{R}^7 , where $\sigma^1, \dots, \sigma^7$ denotes the dual basis of the standard basis and $\sigma^{ij} := \sigma^i \wedge \sigma^j$, $\sigma^{ijk} := \sigma^i \wedge \sigma^j \wedge \sigma^k$. The group $GL(7)$ acts on the space of 3-forms on \mathbb{R}^7 and we define $G_{2(2)} \subset GL(7)$ to be the stabiliser of ω_0 . Then $G_{2(2)}$ is a non-compact group of dimension 14. It is contained in the orthogonal group with respect to the scalar product $2\sigma^1 \cdot \sigma^5 + 2\sigma^2 \cdot \sigma^6 + 2\sigma^3 \cdot \sigma^7 - (\sigma^4)^2$, which has signature $(4, 3)$. It is connected and its fundamental group is \mathbb{Z}_2 . There are other nice characterisations of this group, e.g., $G_{2(2)}$ is the stabiliser of a non-isotropic element of the real spinor representation of $Spin(4, 3)$ and it can also be understood as the stabiliser of a cross product on $\mathbb{R}^{4,3}$. For a Lie group $G \subset O(p, q)$, a G -structure on a pseudo-Riemannian manifold (M, g) of signature (p, q) is a reduction of the bundle $P_{SO(p, q)}$ of orthonormal frames to a G -bundle $P_G \subset P_{SO(p, q)}$. It is called parallel if P_G is parallel with respect to the Levi-Civita connection. The existence of a parallel G -structure is equivalent to the fact that the holonomy group of (M, g) is contained in G . Pseudo-Riemannian symmetric spaces

with parallel G -structures are studied, e.g., for $G = U(r, s) \subset \mathrm{SO}(2r, 2s)$ (pseudo-Hermitian symmetric spaces) in [KO2, KO4], for $G = \mathrm{Sp}(r, s) \subset \mathrm{SO}(4r, 4s)$ (hyper-Kähler symmetric spaces) in [AC1, KO3] and for $G = \mathrm{Sp}(r, s) \cdot \mathrm{Sp}(1) \subset \mathrm{SO}(4r, 4s)$ (quaternionic Kähler case) in [AC2]. Here we turn to $G = \mathrm{G}_{2(2)} \subset \mathrm{SO}(4, 3)$. According to the above remarks on the definition of $\mathrm{G}_{2(2)}$ a parallel $\mathrm{G}_{2(2)}$ -structure can equivalently be defined as a parallel 3-form on M that equals ω_0 with respect to a suitable local frame. Analogously, parallel non-isotropic spinor fields or parallel cross-products can be used to define parallel $\mathrm{G}_{2(2)}$ -structures.

The aim of the paper is to classify all parallel $\mathrm{G}_{2(2)}$ -structures on (simply-connected) indecomposable pseudo-Riemannian symmetric spaces of signature $(4, 3)$. Since any symmetric space (M, g) that admits a $\mathrm{G}_{2(2)}$ -structure is Ricci-flat its transvection group is solvable. In particular, its holonomy group is a proper subgroup of $\mathrm{G}_{2(2)}$ as already remarked above. Since the transvection group is solvable, the symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ that is associated with (M, g) is the quadratic extension of a Lie algebra with involution $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by an orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module \mathfrak{a} as described in [KO2]. Hence we can apply the structure theory for such extensions developed in [KO2].

Henceforth we will use the following notation. If \mathfrak{g} is a Lie algebra and θ is an involution on \mathfrak{g} , then \mathfrak{g}_{\pm} will denote the eigenspace of θ with eigenvalue ± 1 .

Now let (M, g) be a symmetric space with a parallel $\mathrm{G}_{2(2)}$ -structure and $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ the symmetric triple associated with (M, g) . We think of the $\mathrm{G}_{2(2)}$ -structure as a parallel cross product b on (M, g) . Then b corresponds to a \mathfrak{g}_+ -invariant cross product on \mathfrak{g}_- , which we will also denote by b . Up to coverings, (M, g) and its $\mathrm{G}_{2(2)}$ -structure are determined by the symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ and $b : \mathfrak{g}_- \times \mathfrak{g}_- \rightarrow \mathfrak{g}_-$. As remarked above $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ has the structure of a quadratic extension. Moreover, this structure is uniquely determined by the so-called canonical isotropic ideal $\mathfrak{i} \subset \mathfrak{g}$ as explained in [KO2]. Our first step will be to prove that $\mathfrak{i}_- = \mathfrak{i} \cap \mathfrak{g}_-$ is three-dimensional provided the symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ is indecomposable. Then we show that \mathfrak{i}_- is invariant under b . Moreover, after a suitable choice of a section $\mathfrak{l}_- := \mathfrak{g}_- / \mathfrak{i}_-^{\perp} \hookrightarrow \mathfrak{g}_-$, b also restricts to a bilinear map on \mathfrak{l}_- . This will give an additional structure on \mathfrak{l} , which will allow to determine first \mathfrak{l} and then the second quadratic cohomology of \mathfrak{l} with coefficients in $\mathfrak{a} := \mathfrak{i}^{\perp} / \mathfrak{i}$. This cohomology set is in bijection with the equivalence classes of quadratic extensions of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by \mathfrak{a} , which will yield a classification. The final result is a list of all indecomposable symmetric triples with $\mathrm{G}_{2(2)}$ -structure, see Section 6.3. It is formulated in a self-contained manner, i.e., without requiring the knowledge of the results in [KO2]. I would like to thank Martin Olbrich for his interest in this project and for his valuable comments.

2 The spinor representation in signature $(4, 3)$

Let V be an oriented 7-dimensional vector space endowed with a scalar product $\langle \cdot, \cdot \rangle$ of signature $(4, 3)$. Let b_1, \dots, b_7 be a basis of V and denote the dual basis of V^* by $\sigma^1, \dots, \sigma^7$. We will say that b_1, \dots, b_7 is a Witt basis if the scalar product on V equals

$$2\sigma^1 \cdot \sigma^5 + 2\sigma^2 \cdot \sigma^6 + 2\sigma^3 \cdot \sigma^7 - (\sigma^4)^2.$$

The standard example is the vector space \mathbb{R}^7 together with the scalar product $\langle \cdot, \cdot \rangle := -(dx_4)^2 + 2 \sum_{i=1}^3 dx_i dx_{i+4}$, which we will denote by $\mathbb{R}^{4,3}$. The standard basis e_1, \dots, e_7 of \mathbb{R}^7 is a Witt basis of $\mathbb{R}^{4,3}$.

The Clifford algebra \mathcal{C}_V is the unital associative algebra that is generated by the elements of V subject to the relations $uv + vu = -2\langle u, v \rangle$. It is isomorphic to a sum of real matrix algebras, more exactly, $\mathcal{C}_V \cong \text{Mat}(8, \mathbb{R}) \oplus \text{Mat}(8, \mathbb{R})$. This gives us two inequivalent irreducible representations of \mathcal{C}_V on \mathbb{R}^8 . Let $\hat{b}_1, \dots, \hat{b}_7$ be a positively oriented orthonormal basis of V and denote by $\alpha_0 := \hat{b}_1 \dots \hat{b}_7$ the volume element in \mathcal{C}_V . Then the equivalence classes of irreducible representations of \mathcal{C}_V differ by the action of α_0 , which is either the identity or minus identity. If α_0 acts by id we will say that the representation is of type one, otherwise it is of type two. Let Δ_V denote the equivalence class of representations of type one.

Now consider the group

$$\text{Spin}(V) := \langle uv \mid u, v \in V, \langle u, u \rangle = \pm 1, \langle v, v \rangle = \pm 1 \rangle \subset \mathcal{C}_V.$$

If we restrict the two inequivalent representations of \mathcal{C}_V to $\text{Spin}(V)$ we obtain two equivalent irreducible representations of $\text{Spin}(V)$ on \mathbb{R}^8 , which are called spinor representation of $\text{Spin}(V)$. The Lie algebra $\mathfrak{spin}(V)$ of $\text{Spin}(V)$ equals

$$\mathfrak{spin}(V) = \text{span}\{\hat{b}_i \hat{b}_j \mid 1 \leq i < j \leq 7\} \subset \mathcal{C}_V.$$

There is a two-fold covering map $\lambda : \text{Spin}(V) \rightarrow \text{SO}(V)$, which is defined by $\lambda(a)(v) = ava^{-1}$ for $a \in \text{Spin}(V)$ and $v \in V$. In particular, λ induces an isomorphism $\lambda_* : \mathfrak{spin}(V) \rightarrow \mathfrak{so}(V)$. The inverse of this isomorphism is given by

$$\mathfrak{so}(V) \ni A \longmapsto \tilde{A} = \frac{1}{4} \sum_{i=1}^3 \left(b_i A(b_{i+4}) + b_{i+4} A(b_i) \right) - \frac{1}{4} b_4 A(b_4) \in \mathfrak{spin}(V). \quad (2)$$

There exists an inner product $\langle \cdot, \cdot \rangle_\Delta$ of signature $(4, 4)$ on Δ_V satisfying

$$\langle X \cdot \varphi, \psi \rangle_\Delta + \langle \varphi, X \cdot \psi \rangle_\Delta = 0 \quad (3)$$

for all $X \in V$. It is uniquely determined up to multiplication with a real number $\lambda \neq 0$. Independently of the choice of $\langle \cdot, \cdot \rangle_\Delta$ we can speak of isotropic and non-isotropic spinors and of pairs of orthogonal spinors.

Let us give explicit formulas in the case, where $V = \mathbb{R}^{4,3}$. We denote the Clifford algebra of $\mathbb{R}^{4,3}$ by $\mathcal{C}_{4,3}$ and the spin group by $\text{Spin}(4, 3)$. Let s_1, \dots, s_8 denote the standard basis of \mathbb{R}^8 . We define an algebra homomorphism $\Phi : \mathcal{C}_{4,3} \rightarrow \text{Mat}(8, \mathbb{R})$ by

$$\begin{aligned} (1/\sqrt{2}) \cdot \Phi(e_1) &: s_1 \mapsto s_8, s_2 \mapsto s_7, s_3 \mapsto -s_6, s_4 \mapsto -s_5 \\ (1/\sqrt{2}) \cdot \Phi(e_2) &: s_1 \mapsto -s_3, s_2 \mapsto s_4, s_7 \mapsto s_5, s_8 \mapsto -s_6 \\ (1/\sqrt{2}) \cdot \Phi(e_3) &: s_2 \mapsto -s_1, s_4 \mapsto -s_3, s_5 \mapsto s_6, s_7 \mapsto s_8 \\ (1/\sqrt{2}) \cdot \Phi(e_5) &: s_5 \mapsto s_4, s_6 \mapsto s_3, s_7 \mapsto -s_2, s_8 \mapsto -s_1 \\ (1/\sqrt{2}) \cdot \Phi(e_6) &: s_3 \mapsto s_1, s_4 \mapsto -s_2, s_5 \mapsto -s_7, s_6 \mapsto s_8 \\ (1/\sqrt{2}) \cdot \Phi(e_7) &: s_1 \mapsto s_2, s_3 \mapsto s_4, s_6 \mapsto -s_5, s_8 \mapsto -s_7 \\ \Phi(e_4) &: s_i \mapsto s_i, i = 1, 4, 6, 7, \quad s_j \mapsto -s_j, j = 2, 3, 5, 8, \end{aligned} \quad (4)$$

where all basis vectors that are not mentioned are mapped to zero. This will give us a representation of type one of $\mathcal{C}_{4,3}$, which we will denote by $\Delta_{4,3}$. The inner product

$$\langle \cdot, \cdot \rangle_{\Delta} = 2 \sum_{i=1}^4 dx_i dx_{i+4}$$

on $\Delta_{4,3} \cong \mathbb{R}^8$ satisfies (3).

Proposition 2.1 *Let $c \in \mathbb{R}$ be fixed. The group $\text{Spin}(V)$ acts transitively on*

$$\Delta(c) := \{\psi \in \Delta_V \mid \langle \psi, \psi \rangle_{\Delta} = \pm c\}.$$

If $c \neq 0$, then the stabiliser of an element $\psi \in \Delta(c)$ is isomorphic to $G_{2(2)}$.

Furthermore, $\text{Spin}(V)$ acts transitively on

$$\Delta(c, 0) := \{(\psi, \varphi) \in \Delta_V \times \Delta_V \mid \varphi \perp \psi, \langle \psi, \psi \rangle_{\Delta} = \pm c, \langle \varphi, \varphi \rangle_{\Delta} = 0, \varphi \neq 0\}.$$

The Lie algebra $\mathfrak{h}(\psi, \varphi)$ of the stabiliser of an element $(\psi, \varphi) \in \Delta(c, 0)$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathfrak{m}$, where \mathfrak{m} is a central extension of the 3-dimensional Heisenberg algebra by a two-dimensional vector space. If $\mathfrak{a} \subset \mathfrak{h}(\psi, \varphi)$ is an abelian subalgebra, then $\dim \mathfrak{a} \leq 3$.

Proof. The fact that $\text{Spin}(V)$ acts transitively on $\Delta(c)$ with stabiliser isomorphic to $G_{2(2)}$ is well-known, for a proof see, e.g., [K1]. Furthermore, in [K1], Prop. 2.3. it is shown that the unity component $\text{Spin}^+(4, 3)$ of $\text{Spin}(4, 3)$ acts transitively on the Stiefel manifolds

$$V(\varepsilon_1, \varepsilon_2, \varepsilon_3) := \{(\psi_1, \psi_2, \psi_3) \in \Delta_{4,3} \mid \langle \psi_i, \psi_j \rangle_{\Delta} = \varepsilon_i \delta_{ij}\}$$

for $\varepsilon_i = \pm 1$, $i = 1, 2, 3$. This shows that $\text{Spin}(4, 3)$ acts transitively on $\Delta(c, 0)$ since, for a given $(\psi, \varphi) \in \Delta(c, 0)$ we can write $\varphi = \psi_1 + \psi_2$ such that ψ, ψ_1, ψ_2 are orthogonal and $\langle \psi_1, \psi_1 \rangle_{\Delta} = -\langle \psi_2, \psi_2 \rangle_{\Delta} = c$.

The Lie algebra $\mathfrak{h} := \mathfrak{h}(s_1 + s_5, s_6) \subset \mathfrak{spin}(4, 3)$ is spanned by

$$\begin{aligned} &e_1 e_5 - e_2 e_6, \quad e_1 e_6, \quad e_2 e_5, \quad Z_1 := e_1 e_3, \quad Z_2 := e_2 e_3, \\ &N_1 := e_3 e_5 - \sqrt{2} e_2 e_4, \quad N_2 := e_3 e_6 + \sqrt{2} e_1 e_4, \quad N_3 := e_1 e_2 + \sqrt{2} e_3 e_4. \end{aligned}$$

Let us first analyse the structure of the Lie algebra \mathfrak{h} . Note that $e_1 e_6$, $e_2 e_5$ and $e_1 e_5 - e_2 e_6$ span a subalgebra of \mathfrak{h} that is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Moreover, Z_1 , Z_2 , N_1 , N_2 , N_3 span an ideal \mathfrak{m} of \mathfrak{h} , i.e.,

$$\mathfrak{h} = \mathfrak{sl}(2, \mathbb{R}) \ltimes \mathfrak{m}. \quad (5)$$

The centre of \mathfrak{m} equals $\mathfrak{z}(\mathfrak{m}) := \text{span}\{Z_1, Z_2\}$. Hence, as a vector space, \mathfrak{m} is the direct sum of $\mathfrak{z}(\mathfrak{m})$ and $\mathfrak{n} := \text{span}\{N_1, N_2, N_3\}$. Since

$$[N_1, N_2] = -4N_3, \quad [N_1, N_3] = 6Z_2, \quad [N_2, N_3] = -6Z_1 \quad (6)$$

\mathfrak{m} is a central extension of the three-dimensional Heisenberg algebra $\mathfrak{h}(1)$ by $\mathfrak{z}(\mathfrak{m})$. The representation of $\mathfrak{sl}(2, \mathbb{R}) \subset \mathfrak{h}$ on \mathfrak{m} decomposes into a one-dimensional trivial

representation spanned by N_3 and two standard representations spanned by Z_1, Z_2 and N_1, N_2 , respectively. Now let $\mathfrak{a} \subset \mathfrak{h}$ be an abelian subalgebra.

Let us first consider the case where $\mathfrak{a} \subset \mathfrak{m}$. Let $\text{pr}_{\mathfrak{n}}$ denote the projection from $\mathfrak{m} = \mathfrak{z}(\mathfrak{m}) \oplus \mathfrak{n}$ to \mathfrak{n} . Then $\text{pr}_{\mathfrak{n}}(\mathfrak{a})$ is also an abelian subalgebra of \mathfrak{m} . Using (6) we see that $\text{pr}_{\mathfrak{n}}(\mathfrak{a})$ is at most one-dimensional. Hence \mathfrak{a} is at most three-dimensional and $\dim \mathfrak{a} = 3$ holds if and only if $\mathfrak{a} = \mathfrak{z}(\mathfrak{m}) \oplus \mathbb{R} \cdot N$ for some $N \neq 0$ in \mathfrak{n} .

Now suppose $\mathfrak{a} \not\subset \mathfrak{m}$. Since $\mathfrak{sl}(2, \mathbb{R})$ does not contain an abelian subalgebra of dimension two the projection of \mathfrak{a} to $\mathfrak{sl}(2, \mathbb{R})$ with respect to the decomposition (5) is one-dimensional. Thus $\mathfrak{a} = \mathbb{R}(B + M) \oplus \mathfrak{a}_{\mathfrak{m}}$ for some $B \in \mathfrak{sl}(2, \mathbb{R})$ and $M \in \mathfrak{m}$, where $\mathfrak{a}_{\mathfrak{m}}$ is an abelian subalgebra of \mathfrak{m} . We have to show that $\dim \mathfrak{a}_{\mathfrak{m}} \leq 2$ holds. Assume $\dim \mathfrak{a}_{\mathfrak{m}} = 3$. Then the above considerations show $\mathfrak{a}_{\mathfrak{m}} = \mathfrak{z}(\mathfrak{m}) \oplus \mathbb{R} \cdot N$ for some $N \neq 0$ in \mathfrak{n} . Since \mathfrak{a} is abelian $B \neq 0$ has to act trivially on $\mathfrak{z}(\mathfrak{m}) \subset \mathfrak{a}_{\mathfrak{m}}$, which is a contradiction. \square

Proposition 2.2 1. If $\psi \in \Delta_V$ is non-isotropic, then $V \ni X \mapsto X \cdot \psi \in \psi^\perp$ is an isomorphism.

2. The map

$$\Delta_V \ni \varphi \longmapsto U_\varphi := \{X \in V \mid X \cdot \varphi = 0\} \subset V$$

induces a bijection from the set $\{\varphi \in \Delta_V \mid \langle \varphi, \varphi \rangle = 0, \varphi \neq 0\} / \mathbb{R}$ of projective isotropic spinors to the set of 3-dimensional isotropic subspaces of V . If $\varphi \in \Delta_V$ is isotropic, then $U_\varphi^\perp \cdot \varphi \subset \mathbb{R} \cdot \varphi$.

Proof. By Prop. 2.1 we may assume $\psi = s_1 + s_5 \in \Delta_{4,3}$ and $\varphi = s_6 \in \Delta_{4,3}$ and the assertion follows from Equation (4). \square

Definition 2.3 For the time being, consider V without orientation. A 3-form ω on V is called nice if there is a Witt basis b_1, \dots, b_7 of V such that

$$\omega = \sqrt{2}(\sigma^{127} + \sigma^{356}) - \sigma^4 \wedge (\sigma^{15} + \sigma^{26} - \sigma^{37}). \quad (7)$$

with respect to the dual basis $\sigma^1, \dots, \sigma^7$.

The stabiliser of a nice 3-form ω is isomorphic to $G_{2(2)}$. In particular, ω induces an orientation on V , since $G_{2(2)}$ is connected.

Definition 2.4 Consider again V without orientation. A bilinear map $b : V \times V \rightarrow V$ is called a cross product if

- (i) $b(X, Y) = -b(Y, X)$,
- (ii) $\langle X, b(X, Y) \rangle = 0$,
- (iii) $b(X, b(X, Y)) = -\langle X, X \rangle Y + \langle X, Y \rangle X$.

The following proposition is proven in [K1].

Proposition 2.5 1. *The map*

$$\mathcal{B} \longrightarrow \Lambda_{\sharp}^3(V^*), \quad b \longmapsto \omega_b := \langle \cdot, b(\cdot, \cdot) \rangle$$

is a bijection between the set \mathcal{B} of cross products and the set $\Lambda_{\sharp}^3(V^)$ of nice 3-forms on V .*

2. *Now let V be oriented and denote by \mathcal{B}^+ the set of cross products b for which the orientation induced by ω_b coincides with the orientation of V . Then the map*

$$\Delta_* := \{\psi \in \Delta_V \mid \langle \psi, \psi \rangle_{\Delta} \neq 0\} \longrightarrow \mathcal{B}^+, \quad \psi \longmapsto b_{\psi}$$

defined by

$$XY \cdot \psi + \langle X, Y \rangle \psi = b_{\psi}(X, Y) \cdot \psi$$

induces a bijection from the set $P(\Delta_) := \Delta_*/\mathbb{R}$ of projective non-isotropic spinors in Δ_V to \mathcal{B}^+ .*

If we consider, in particular, $\psi = s_1 + s_5 \in \Delta_{4,3}$, then the 3-form that is associated with ψ according to Proposition 2.5 equals ω_0 as defined in (1).

3 $G_{2(2)}$ -structures on symmetric spaces

3.1 Symmetric spaces and symmetric triples

Before we start let us introduce the following convention. If \mathfrak{g} is a Lie algebra and θ is an involutive automorphism on \mathfrak{g} , then we denote the eigenspaces of θ with eigenvalues 1 and -1 by \mathfrak{g}_+ and \mathfrak{g}_- , respectively.

Let M be a (pseudo-Riemannian) symmetric space and choose a base point $u \in M$. Then $M = G/G_+$, where G is the transvection group of M and $G_+ \subset G$ is the stabiliser of $u \in M$. The conjugation by the reflection of M at u induces an involution on G and therefore also on \mathfrak{g} . We denote this involution by θ . If $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ is the decomposition of \mathfrak{g} into eigenspaces of θ , then \mathfrak{g}_+ is the Lie algebra of G_+ and the vector space \mathfrak{g}_- can be identified with $T_u M$. Moreover, G_+ equals the holonomy group of (M, g) and its adjoint representation on \mathfrak{g}_- is the holonomy representation. The scalar product on $\mathfrak{g}_- \cong T_u M$ has a unique extension to an $\text{ad}(\mathfrak{g})$ -invariant non-degenerate inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} such that $\mathfrak{g}_+ \perp \mathfrak{g}_-$. In particular, the triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ consists of a metric Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ and an isometric involution $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$. Moreover, $[\mathfrak{g}_-, \mathfrak{g}_-] = \mathfrak{g}_+$. Any triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ satisfying these properties will be called a symmetric triple. The above described assignment of a symmetric triple to a symmetric space gives a bijection from the set of simply-connected symmetric spaces to the set of symmetric triples. Isometry classes of simply-connected symmetric spaces correspond to isomorphism classes of symmetric triples. Furthermore, a symmetric space is indecomposable if and only if the associated symmetric triple is indecomposable, i.e., if it is not a non-trivial direct sum of two symmetric triples.

The signature of a symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ is defined as the signature of $\langle \cdot, \cdot \rangle$ restricted to \mathfrak{g}_- , which equals the signature of M .

3.2 $G_{2(2)}$ -structures on symmetric spaces and symmetric triples

Definition 3.1 A $G_{2(2)}$ -structure on a pseudo-Riemannian manifold (M, g) of signature $(4, 3)$ is a section $\omega \in \Omega^3(M)$ such that ω_x is a nice 3-form on $T_x M$ for each $x \in M$.

Definition 3.2 A $G_{2(2)}$ -structure ω on a symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ of signature $(4, 3)$ is a nice \mathfrak{g}_+ -invariant 3-form ω on \mathfrak{g}_- .

According to Prop. 2.5, it can equivalently be considered as a \mathfrak{g}_+ -invariant cross product b on \mathfrak{g}_- or as a pair $(\mathcal{O}, [\psi])$, where \mathcal{O} is an orientation on \mathfrak{g}_- , $\psi \in \Delta_{\mathfrak{g}_-}$ is a \mathfrak{g}_+ -invariant non-isotropic element of the representation $\Delta_{\mathfrak{g}_-}$ of $\mathcal{C}(\mathfrak{g}_-)$ of type one and $[\psi]$ is the projective spinor represented by ψ .

Definition 3.3 Two symmetric triples with $G_{2(2)}$ -structure $(\mathfrak{g}_i, \theta_i, \omega_i, \langle \cdot, \cdot \rangle_i)$, $i = 1, 2$, are called isomorphic if there is an isomorphism $\phi : (\mathfrak{g}_1, \theta_1, \langle \cdot, \cdot \rangle_1) \rightarrow (\mathfrak{g}_2, \theta_2, \langle \cdot, \cdot \rangle_2)$ of symmetric triples satisfying $\phi * \omega_2 = \omega_1$.

The following proposition is a consequence of the holonomy principle.

Proposition 3.4 There is a 1-1-correspondence between parallel $G_{2(2)}$ -structures on (M, g) and $G_{2(2)}$ -structures on the associated symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$.

Since the stabiliser of a nice 3-form is isomorphic to $G_{2(2)}$ the existence of a parallel $G_{2(2)}$ -structure implies that the holonomy group is contained in $G_{2(2)}$.

Proposition 3.5 If a symmetric space M admits a parallel $G_{2(2)}$ -structure, then its transvection group is solvable.

Proof. If a pseudo-Riemannian manifold of signature $(4, 3)$ has a parallel $G_{2(2)}$ -structure, then it admits a spin structure and a parallel non-isotropic spinor field ψ . In particular, $\text{Ric}(X) \cdot \psi = 0$ holds for any vector field X , see, e.g., [BFGK], hence M is Ricci-flat. On the other hand, the Ricci tensor on $T_u M \cong \mathfrak{g}_-$ is given by the Killing form $\kappa_{\mathfrak{g}}$, more exactly, $\text{Ric}(X, Y) = -(1/2) \cdot \kappa_{\mathfrak{g}}(X, Y)$. Hence, $\kappa_{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{g}_-) = 0$. Moreover, $\kappa_{\mathfrak{g}}(\mathfrak{g}_+, \mathfrak{g}_+) = \kappa_{\mathfrak{g}}([\mathfrak{g}_-, \mathfrak{g}_-], \mathfrak{g}_+) \subset \kappa_{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{g}_-) = 0$. Furthermore, $\kappa_{\mathfrak{g}}(\mathfrak{g}_+, \mathfrak{g}_-) = 0$ since $\kappa_{\mathfrak{g}}$ is invariant under automorphisms, in particular, under θ . Thus $\kappa_{\mathfrak{g}} = 0$. Consequently, \mathfrak{g} is solvable by Cartan's first criterion. \square

In particular, this shows that the holonomy group of a symmetric space with $G_{2(2)}$ -structure is solvable, thus it is always properly contained in $G_{2(2)}$. Here we want to assume, that the holonomy group does not become 'too small'. More exactly, we will consider only indecomposable symmetric spaces with $G_{2(2)}$ -structure.

4 Quadratic extensions and $G_{2(2)}$ -structures

4.1 Quadratic extensions

Since the existence of a $G_{2(2)}$ -structure on a symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ implies that \mathfrak{g} is solvable, the Lie algebra \mathfrak{g} does not have simple ideals, thus the theory of quadratic

extension applies. This theory is developed in [KO2]. There it is proven that any symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ without simple ideals has the structure of a canonically determined admissible quadratic extension of some proper Lie algebra with involution $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by an orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module $\mathfrak{a} := (\rho, \alpha, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, \theta_{\mathfrak{a}})$. Here the condition *proper* for $(\mathfrak{l}, \theta_{\mathfrak{l}})$ means that $[\mathfrak{l}_-, \mathfrak{l}_-] = \mathfrak{l}_+$ holds. Moreover, any admissible quadratic extension of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by \mathfrak{a} is equivalent to one of the following kind called standard model. It is denoted by $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ and is obtained in the following way. As a symmetric triple it equals $(\mathfrak{d}, \theta, \langle \cdot, \cdot \rangle)$, where \mathfrak{d}, θ and $\langle \cdot, \cdot \rangle$ are defined as follows. Assume that $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})_+$ is an admissible cocycle, i.e., $[\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_{\#} \subset \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$. Note that this assumption includes the condition that the representation of \mathfrak{l} on \mathfrak{a} is semisimple. Then \mathfrak{d} equals the vector space $\mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l}$, the inner product $\langle \cdot, \cdot \rangle$ and an involutive endomorphism θ on \mathfrak{d} are given by

$$\begin{aligned} \langle Z + A + L, Z' + A' + L' \rangle &:= \langle A, A' \rangle_{\mathfrak{a}} + Z(L') + Z'(L) \\ \theta(Z + A + L) &:= \theta_{\mathfrak{l}}^*(Z) + \theta_{\mathfrak{a}}(A) + \theta_{\mathfrak{l}}(L) \end{aligned}$$

for $Z, Z' \in \mathfrak{l}^*$, $A, A' \in \mathfrak{a}$ and $L, L' \in \mathfrak{l}$. Furthermore, the Lie bracket $[\cdot, \cdot] : \mathfrak{d} \times \mathfrak{d} \rightarrow \mathfrak{d}$ is defined by $[\mathfrak{l}^*, \mathfrak{l}^* \oplus \mathfrak{a}] = 0$ and

$$\begin{aligned} [L, L'] &= \gamma(L, L', \cdot) + \alpha(L, L') + [L, L']_{\mathfrak{l}} \\ [L, A] &= \rho(L)(A) - \langle A, \alpha(L, \cdot) \rangle \\ [L, Z] &= \text{ad}^*(L)(Z) \\ [A, A'] &= \langle \rho(\cdot)(A), A' \rangle \end{aligned} \tag{8}$$

for $Z \in \mathfrak{l}^*$, $A, A' \in \mathfrak{a}$ and $L, L' \in \mathfrak{l}$. We identify the vector space $\mathfrak{d}/\mathfrak{l}^*$ with $\mathfrak{a} \oplus \mathfrak{l}$ and denote by $i : \mathfrak{a} \rightarrow \mathfrak{a} \oplus \mathfrak{l}$ the injection and by $p : \mathfrak{a} \oplus \mathfrak{l} \rightarrow \mathfrak{l}$ the projection. Then $(\mathfrak{d}, \mathfrak{l}^*, i, p)$ is an admissible quadratic extension of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by \mathfrak{a} . We denote this quadratic extension as well as the underlying symmetric triple by $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$.

Recall that for any metric Lie algebra \mathfrak{g} the canonical isotropic ideal is defined [KO1]. Since the considered quadratic extension $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ is admissible the canonical isotropic ideal \mathfrak{i} coincides with \mathfrak{l}^* .

For admissible cocycles (α_1, γ_1) and (α_2, γ_2) in $\mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})_+$ the quadratic extensions $\mathfrak{d}_{\alpha_i, \gamma_i}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$, $i = 1, 2$, are equivalent if and only if $[\alpha_1, \gamma_1] = [\alpha_2, \gamma_2] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_{\#}$.

Let $(\mathfrak{l}_i, \theta_{\mathfrak{l}_i})$, $i = 1, 2$, be Lie algebras with involution and let \mathfrak{a}_i be orthogonal $(\mathfrak{l}_i, \theta_{\mathfrak{l}_i})$ -modules. An isomorphism of triples $(S, U) : (\mathfrak{l}_1, \theta_{\mathfrak{l}_1}, \mathfrak{a}_1) \rightarrow (\mathfrak{l}_2, \theta_{\mathfrak{l}_2}, \mathfrak{a}_2)$ consists of an isomorphism $S : (\mathfrak{l}_1, \theta_{\mathfrak{l}_1}) \rightarrow (\mathfrak{l}_2, \theta_{\mathfrak{l}_2})$ of Lie algebras with involution and an isometry $U : \mathfrak{a}_2 \rightarrow \mathfrak{a}_1$ such that $U \circ \rho_2(S(L)) = \rho_1(L) \circ U$ holds for all $L \in \mathfrak{l}_1$.

Let $(\mathfrak{l}, \theta_{\mathfrak{l}})$ and $(\mathfrak{l}', \theta_{\mathfrak{l}'})$ be proper Lie algebras with involution. For $[\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_{\#}$ and $[\alpha', \gamma'] \in \mathcal{H}_Q^2(\mathfrak{l}', \theta_{\mathfrak{l}'}, \mathfrak{a}')_{\#}$ the symmetric triples $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ and $\mathfrak{d}_{\alpha', \gamma'}(\mathfrak{l}', \theta_{\mathfrak{l}'}, \mathfrak{a}')$ are isomorphic if and only if there is an isomorphism of triples $(S, U) : (\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}) \rightarrow (\mathfrak{l}', \theta_{\mathfrak{l}'}, \mathfrak{a}')$ such that $(S, U)^*[\alpha', \gamma'] = [\alpha, \gamma]$.

Proposition 4.1 *A symmetric space (M, g) that is associated with a quadratic extension of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by \mathfrak{a} is Ricci-flat if and only if the trace form of the representation ρ of \mathfrak{l} on \mathfrak{a} and the Killing form $\kappa_{\mathfrak{l}}$ of \mathfrak{l} are related by $t_{\rho} = -2\kappa_{\mathfrak{l}}$.*

Proof. In the proof of Prop. 3.5 we have already seen that (M, g) is Ricci-flat if and only if $\kappa_{\mathfrak{g}} = 0$ holds for the associated symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$. If \mathfrak{g} is isomorphic to a quadratic extension of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by \mathfrak{a} then $\kappa_{\mathfrak{g}}(X_1, X_2) = 0$ for $X_1, X_2 \in \mathfrak{l}^* \oplus \mathfrak{a}$ and $\kappa_{\mathfrak{g}}(L_1, L_2) = 2\kappa_{\mathfrak{l}}(L_1, L_2) + t_{\rho}(L_1, L_2)$ for all $L_1, L_2 \in \mathfrak{l}$ by Equations (8). \square

4.2 The dimension of \mathfrak{l}_-

We already know that any symmetric triple that admits a $G_{2(2)}$ -structure is a quadratic extension of a Lie algebra with involution $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by an orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module \mathfrak{a} . Now let $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ be a symmetric triple of signature $(4, 3)$ that is isomorphic to the admissible quadratic extension $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$. In this section we will prove, that the existence of a $G_{2(2)}$ -structure on $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ implies $\dim \mathfrak{l}_- = 3$ provided that the symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ is indecomposable.

Lemma 4.2 *If $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ is indecomposable and has a $G_{2(2)}$ -structure, then $\dim \mathfrak{l}_- < 3$ implies $\dim \mathfrak{g}_+ \leq 3$.*

Proof. If $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ admits a $G_{2(2)}$ -structure, then \mathfrak{g} is solvable. By Lie's theorem $[\mathfrak{g}, \mathfrak{g}]$ acts on \mathfrak{g} by nilpotent endomorphisms. Thus $\mathfrak{g}_+ \subset [\mathfrak{g}, \mathfrak{g}]$ acts nilpotently on \mathfrak{g}_- . Hence there exists an element $U \neq 0$ in \mathfrak{g}_- such that $[\mathfrak{g}_+, U] = 0$. Since \mathfrak{g} is indecomposable U is isotropic. Now let $(\mathcal{O}, [\psi])$ be a $G_{2(2)}$ -structure. Then $U \cdot \psi \in \Delta_{\mathfrak{g}_-}$ is a \mathfrak{g}_+ -invariant isotropic spinor and $\psi \perp U \cdot \psi$. Now Corollary 2.1 implies that the maximal dimension of an abelian Lie algebra contained in \mathfrak{g}_+ is three.

Finally, we will prove that \mathfrak{g}_+ is abelian, which will imply the assertion of the lemma. The metric Lie algebra \mathfrak{g}_+ is a quadratic extension of \mathfrak{l}_+ by \mathfrak{a}_+ . In our situation $\dim \mathfrak{l}_+ \leq 1$. Moreover, $\rho|_{\mathfrak{l}_+} = 0$ by Lie's theorem since \mathfrak{l} is solvable, $\mathfrak{l}_+ = [\mathfrak{l}_-, \mathfrak{l}_-] \subset [\mathfrak{l}, \mathfrak{l}]$ and ρ is semisimple. Thus \mathfrak{g}_+ is abelian. \square

Proposition 4.3 *If $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ is indecomposable and admits a $G_{2(2)}$ -structure, then $\dim \mathfrak{l}_- = 3$.*

Proof. If $\dim \mathfrak{l}_- < 3$, then $\dim \mathfrak{l}_+ \leq 1$. Let us first consider the case $\dim \mathfrak{l}_+ = 0$. In this case \mathfrak{l} is abelian, hence $\mathfrak{l} = \mathbb{R}$ or $\mathfrak{l} = \mathbb{R}^2$. If $\mathfrak{l} = \mathbb{R}$, then $\alpha = \gamma = 0$. Thus $\mathfrak{a} = \rho(\mathfrak{l})(\mathfrak{a})$ since \mathfrak{g} is indecomposable. Because of $\dim \mathfrak{a}_- = 7 - 2 \cdot \dim \mathfrak{l}_- = 5$ this shows that \mathfrak{a}_+ has also dimension 5. Hence $\mathfrak{g}_+ = \mathfrak{a}_+$ is a five-dimensional abelian Lie algebra, which is a contradiction to Lemma 4.2.

If $\mathfrak{l} = \mathbb{R}^2 = \text{span}\{Y, Z\}$, then \mathfrak{a}_- has signature $(2, 1)$. First consider the case $\alpha = 0$. Every non-trivial indecomposable orthogonal \mathbb{R}^2 -module $(\bar{\mathfrak{a}}, \bar{\rho})$ is of one of the following types

- (i) $\bar{\mathfrak{a}} \in \{\mathbb{R}^{2,0}, \mathbb{R}^2\}$, $\bar{\rho}_{\mathbb{C}}$ has weights $\pm i\lambda$, $\lambda \in \mathfrak{l}^*$,
- (ii) $\bar{\mathfrak{a}} = \mathbb{R}^{1,1}$, $\bar{\rho}_{\mathbb{C}}$ has weights $\pm \lambda$, $\lambda \in \mathfrak{l}^*$,
- (iii) $\bar{\mathfrak{a}} = \mathbb{R}^{2,2}$, where $\mathfrak{a}_+ = \mathbb{R}^{1,1}$, $\mathfrak{a}_- = \mathbb{R}^{1,1}$ and $\bar{\rho}_{\mathbb{C}}$ has weights $\pm \mu \pm i\nu$, $\mu, \nu \in \mathfrak{l}^*$.

Since ρ does not contain a trivial subrepresentation and $t_\rho = 0$ by Prop. 4.1 we have the following possibilities for (\mathfrak{a}, ρ) :

(a) \mathfrak{a} is the sum of two representations of type (i) with weights $\pm i\lambda_1$ and $\pm i\lambda_2$ and one representation of type (ii) with weights $\pm\lambda_3$. Then $t_\rho = 0$ gives

$$-\lambda_1(L)\lambda_1(L') - \lambda_2(L)\lambda_2(L') + \lambda_3(L)\lambda_3(L') = 0. \quad (9)$$

Put $\lambda := (\lambda_1, \lambda_2, \lambda_3)$. Then (9) says that $\lambda(Y)$ and $\lambda(Z)$ span an isotropic subspace of a pseudo-Euclidean space of signature $(2, 1)$. The maximal dimension of such a subspace is one. Thus $\lambda(Y)$ and $\lambda(Z)$ are linearly dependent, which contradicts the indecomposability of \mathfrak{g} .

(b) \mathfrak{a} is the sum of two representations of type (ii) and one representation of type (i). Then the argumentation is as in case (a).

(c) \mathfrak{a} is the sum of a representation of type (iii) and a representation of type (i) or (ii). Then

$$\pm\lambda(L)\lambda(L') + 4\mu(L)\mu(L') - 4\nu(L)\nu(L') = 0$$

holds for all $L, L' \in \mathfrak{l}$. As in (a) we see that $(\lambda(Y), \mu(Y), \nu(Y))$ and $(\lambda(Z), \mu(Z), \nu(Z))$ are linearly dependent, which contradicts indecomposability.

Now we consider the case $\alpha \neq 0$. Then the $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module \mathfrak{a} decomposes into the one-dimensional image of α and a six-dimensional $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module \mathfrak{a}' , which is equivalent to the module \mathfrak{a} considered in the case $\alpha = 0$. Hence $\mathfrak{g}_+ = \mathfrak{a}_+$ is four-dimensional. Thus \mathfrak{g} cannot obtain a $G_{2(2)}$ -structure by Lemma 4.2.

Now assume $\dim \mathfrak{l}_+ = 1$. Then $\dim \mathfrak{l}_- = 2$ and $\mathfrak{l} \in \{\mathfrak{h}(1), \mathfrak{n}(2), \mathfrak{r}_{3,-1}\}$, see [KO2], Prop. 7.2. Moreover, \mathfrak{a}_- has signature $(2, 1)$. Since ρ is semisimple and \mathfrak{l} is solvable ρ is a representation of the abelian Lie algebra $\mathfrak{l}/[\mathfrak{l}, \mathfrak{l}]$. We decompose $\mathfrak{a} = \mathfrak{a}^{\mathfrak{l}} \oplus \mathfrak{a}'$ with $\mathfrak{a}' = \rho(\mathfrak{l})(\mathfrak{a})$. Since $\mathfrak{a}^{\mathfrak{l}} \subset \alpha(\mathfrak{l}, \mathfrak{l})$ we have $\dim \mathfrak{a}_-^{\mathfrak{l}} \leq 2$. Thus $\dim \mathfrak{a}_-^{\mathfrak{l}} \geq 1$. If $\dim \mathfrak{a}_+^{\mathfrak{l}} = \dim \mathfrak{a}_-^{\mathfrak{l}} > 1$, then we are done. Hence assume $\dim \mathfrak{a}_-^{\mathfrak{l}} = 2$ and $\dim \mathfrak{a}_+^{\mathfrak{l}} = \dim \mathfrak{a}_-^{\mathfrak{l}} = 1$. The latter equation implies that $\mathfrak{l} \neq \mathfrak{h}(1)$ since the Killing form of $\mathfrak{h}(1)$ vanishes, which would imply $t_\rho = 0$ by Prop. 4.1, thus $\rho = 0$, a contradiction. Furthermore, $\dim \mathfrak{a}_-^{\mathfrak{l}} = 2$ implies that $\mathfrak{l} \notin \{\mathfrak{n}(2), \mathfrak{r}_{3,-1}\}$, since in both cases $\mathfrak{a}_-^{\mathfrak{l}}$ is spanned by $\alpha(Y, Z)$ for any $[\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_0$, see [KO2], Prop. 7.3 (note that there is an obvious typo). \square

4.3 The canonical isotropic ideal and the cross product

Let $(\mathfrak{g}, \theta, \omega, \langle \cdot, \cdot \rangle)$ be a symmetric triple with $G_{2(2)}$ -structure and let b be the cross product that corresponds to ω . We will prove that the canonical isotropic ideal $\mathfrak{i}_- = \mathfrak{i} \cap \mathfrak{g}_-$ is invariant under b . To this end we consider the $G_{2(2)}$ -structure ω equivalently as a pair $(\mathcal{O}, [\psi])$ according to the remark after Definition 3.2. Furthermore, let $\varphi \in \Delta_{\mathfrak{g}_-}$, $\varphi \neq 0$, be an isotropic spinor corresponding to $U_\varphi = \mathfrak{i}_-$, i.e., $\mathfrak{i}_- \cdot \varphi = 0$. Recall that φ is uniquely determined up to multiplication with a real number $r \neq 0$.

Lemma 4.4 *The spinor φ is \mathfrak{g}_+ -invariant.*

Proof. Let $X \in \mathfrak{g}_+$ be arbitrary. Note that $Y \cdot A \cdot \varphi = A \cdot Y \cdot \varphi - \lambda_*(A)(Y) \cdot \varphi$ holds for all $Y \in \mathfrak{g}_-$ and $A \in \mathfrak{spin}(\mathfrak{g}_-)$. In particular, take $Y \in U_\varphi$. Then

$$Y \cdot \widetilde{\text{ad}}(X) \cdot \varphi = \widetilde{\text{ad}}(X) \cdot Y \cdot \varphi - [X, Y] \cdot \varphi = -[X, Y] \cdot \varphi.$$

Since $U_\varphi = \mathfrak{i}_-$ is \mathfrak{g}_+ -invariant, we have $[X, Y] \in U_\varphi$ and thus $Y \cdot \widetilde{\text{ad}}(X) \cdot \varphi = 0$ for all $Y \in U_\varphi$. Hence $\widetilde{\text{ad}}(X) \cdot \varphi = t\varphi$ for some $t \in \mathbb{R}$. On the other hand, $X \in \mathfrak{g}_+ \subset [\mathfrak{g}, \mathfrak{g}]$ acts nilpotently on $\Delta_{\mathfrak{g}_-}$ since \mathfrak{g} is solvable. Hence $t = 0$, which proves the claim. \square

Lemma 4.5 *If \mathfrak{g} is indecomposable, then $\varphi \perp \psi$.*

Proof. Assume that $\langle \varphi, \psi \rangle_\Delta \neq 0$. Then we can choose \mathfrak{g}_+ -invariant orthogonal elements $\psi_1, \psi_2 \in \Delta_{\mathfrak{g}_-}$ such that $\langle \psi_i, \psi_i \rangle_\Delta \neq 0$, $i = 1, 2$. By Proposition 2.2 we can define a vector X in \mathfrak{g}_- such that $\psi_2 = X \cdot \psi_1$. Then X is not isotropic and satisfies $[\mathfrak{g}_+, X] = 0$. This is a contradiction to the indecomposability of \mathfrak{g} . \square

Proposition 4.6 *The cross product b has the following properties*

- (i) $b(\mathfrak{i}_-^\perp, \mathfrak{i}_-^\perp) \subset \mathfrak{i}_-$,
- (ii) $\mathfrak{n}^* := b(\mathfrak{i}_-, \mathfrak{i}_-)^\perp$ satisfies $\mathfrak{n}^* \neq 0$, $b(\mathfrak{n}^*, \mathfrak{i}_-) = 0$ and $b(\mathfrak{n}^*, \mathfrak{i}_-^\perp) = \mathfrak{n}^*$.

Proof. Any oriented (with respect to the orientation induced by ω) Witt basis $\mathbf{b} = (b_1, \dots, b_7)$ of \mathfrak{g}_- gives us an isometry from \mathfrak{g}_- to $\mathbb{R}^{4,3}$. Thus we can identify also $\Delta_{\mathfrak{g}_-} \cong \Delta_{4,3}$. By Prop. 2.1 we can choose \mathbf{b} in such a way that $\psi = s_1 + s_5$ and $\varphi = s_6$. Then $\mathfrak{i}_- = U_\varphi = \text{span}\{b_1, b_2, b_3\}$ and $\mathfrak{i}_-^\perp = \text{span}\{b_4, \dots, b_7\}$. Furthermore, ω is equal to the 3-form ω_0 , which was defined by Equation (1). Hence

$$\begin{aligned} b(b_1, b_2) &= \sqrt{2}b_3, \quad b(b_1, b_3) = b(b_2, b_3) = 0 \\ b(b_1, b_4) &= -b_1, \quad b(b_2, b_4) = -b_2, \quad b(b_3, b_4) = b_3 \end{aligned}$$

thus $\mathfrak{n}^* = \mathbb{R} \cdot b_3$ and the assertion follows. \square

Using Prop. 4.6, (ii) we can define an orientation of the one-dimensional vector space $\mathfrak{i}_-^\perp/\mathfrak{i}_-$ in the following way. Note first that $b : \mathfrak{i}_-^\perp/\mathfrak{i}_- \otimes \mathfrak{n}^* \rightarrow \mathfrak{n}^*$ is correctly defined.

Definition 4.7 *A vector A in $\mathfrak{i}_-^\perp/\mathfrak{i}_-$ is said to have a positive orientation if $b(A, U)$ is a positive multiple of U for all $U \in \mathfrak{n}^*$.*

Definition 4.8 *A Lie algebra with B-structure $(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}})$ consists of a solvable Lie algebra with involution $(\mathfrak{l}, \theta_{\mathfrak{l}})$ satisfying $[\mathfrak{l}_-, \mathfrak{l}_-] = \mathfrak{l}_+$ and a non-trivial antisymmetric bilinear map $b_{\mathfrak{m}} : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{l}_-$ defined on a two-dimensional subspace \mathfrak{m} of \mathfrak{l}_- such that*

- (L1) $[\mathfrak{l}_+, \mathfrak{m}] \subset \mathfrak{m}$,
- (L2) $[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{l}} = 0$,
- (L3) $\mathfrak{n} := b_{\mathfrak{m}}(\mathfrak{m}, \mathfrak{m})$ is complementary to \mathfrak{m} in \mathfrak{l}_- .

An isomorphism of Lie algebras with B-structure $S : (\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}) \rightarrow (\mathfrak{l}', \theta_{\mathfrak{l}'}, b_{\mathfrak{m}'})$ is an isomorphism of Lie algebras with involutions that satisfies in addition $S(\mathfrak{m}) = \mathfrak{m}'$ and

$$b_{\mathfrak{m}'}(SL, SL') = Sb_{\mathfrak{m}}(L, L')$$

holds for all $L, L' \in \mathfrak{m}$.

Remark 4.9 Since \mathfrak{l} is solvable and $\mathfrak{l}_+ \subset [\mathfrak{l}, \mathfrak{l}]$ the operator $\text{ad}(L_+)$ is nilpotent for all $L_+ \in \mathfrak{l}_+$. Hence (L1) implies

$$[\mathfrak{l}_+, \mathfrak{l}_-] \subset \mathfrak{m} \quad (10)$$

and $\text{tr ad}(L_+)|_{\mathfrak{m}} = 0$ for all $L_+ \in \mathfrak{l}_+$. The latter equation is equivalent to

$$b_{\mathfrak{m}}([L_+, L_1]_{\mathfrak{l}}, L_2) + b_{\mathfrak{m}}(L_1, [L_+, L_2]_{\mathfrak{l}}) = 0. \quad (11)$$

Indeed, the left hand side of (11) equals $\text{tr ad}(L_+)|_{\mathfrak{m}} \cdot b_{\mathfrak{m}}(L_1, L_2)$.

Let $(\mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, A)$ be a pseudo-Euclidean space with distinguished time-like unit vector A , i.e., $\langle A, A \rangle_{\mathfrak{a}} = -1$. We define an involution $\theta_{\mathfrak{a}}$ on \mathfrak{a} by

$$\mathfrak{a}_- = \mathbb{R} \cdot A, \quad \mathfrak{a}_+ = \mathfrak{a}_-^{\perp}. \quad (12)$$

We will call $\theta_{\mathfrak{a}}$ the induced involution. The vector A may be considered as an orientation of \mathfrak{a}_- .

5 The standard model

5.1 Quadratic cocycles of Lie algebras with B -structure

Let $(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}})$ be a Lie algebra with B -structure and $(\mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, A)$ a pseudo-Euclidean space with distinguished time-like unit vector A . Let $\theta_{\mathfrak{a}}$ be the induced involution as defined in (12). We consider \mathfrak{a} as a trivial $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module.

Definition 5.1 We define $\mathcal{Z}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$ to be the set of those $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})_+$ that satisfy

$$(Z1) \quad \alpha(\mathfrak{m}, \mathfrak{m}) = 0;$$

$$(Z2) \quad [L, b_{\mathfrak{m}}(L', L'')] = \langle \alpha(L, L''), A \rangle L' - \langle \alpha(L, L'), A \rangle L'' \text{ for all } L \in \mathfrak{l}_+, L', L'' \in \mathfrak{m};$$

$$(Z3) \quad 2\gamma(L, L', L'') = -\langle A, \alpha(L, b_{\mathfrak{m}}(L', L'')) \rangle \text{ for all } L \in \mathfrak{l}_+, L', L'' \in \mathfrak{m}.$$

We will say that an element of $\mathcal{Z}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$ is admissible if it is admissible as an element of $\mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})_+$ and we will denote by $\mathcal{Z}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})_{\sharp}$ the set of admissible elements of $\mathcal{Z}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$.

In the following we will need the group

$$\mathcal{N} := \{S_0 \in \text{Aut}(\mathfrak{l}, \theta_{\mathfrak{l}}) \mid S_0(\mathfrak{m}) \subset \mathfrak{m}, S_0|_{\mathfrak{m}} = \text{id}_{\mathfrak{m}}, \bar{S}_0 = \text{id}_{\mathfrak{l}_-/\mathfrak{m}}, S_0|_{\mathfrak{l}_+} = \text{id}\}, \quad (13)$$

where \bar{S}_0 denotes the map induced by S_0 on $\mathfrak{l}_-/\mathfrak{m}$.

Definition 5.2 We will say that elements (α, γ) and (α', γ') of $\mathcal{Z}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$ are equivalent if and only if there exist an isomorphism $S_0 \in \mathcal{N}$ and an element $(\tau, \sigma) \in \mathcal{C}_Q^1(\mathfrak{l}, \mathfrak{a})_+$ such that

(i) (τ, σ) has the properties

- (B1) $\tau(L) = \text{tr}(\text{pr}_{\mathfrak{m}} S_0(b_{\mathfrak{m}}(L, \cdot))) \cdot A$ for all $L \in \mathfrak{m}$,
where $\text{pr}_{\mathfrak{m}} : \mathfrak{l} = \mathfrak{m} \oplus \mathfrak{n} \rightarrow \mathfrak{m}$ is the projection,
(B2) $2\sigma(L', L'') = -\langle \tau(b_{\mathfrak{m}}(L', L'')), A \rangle$ for all $L', L'' \in \mathfrak{m}$;

(ii) $(\alpha, \gamma) = (S_0^* \alpha', S_0^* \gamma') \cdot (\tau, \sigma)$.

We will denote the set of equivalence classes by $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$.

5.2 Construction of the standard model

Let $(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}})$ be a Lie algebra with B -structure. Recall that, in particular, $(\mathfrak{l}, \theta_{\mathfrak{l}})$ is proper. Let $(\mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, A)$ be a pseudo-Euclidean space with distinguished time-like unit vector A . Let $\theta_{\mathfrak{a}}$ be the induced involution as defined in (12). As above we consider \mathfrak{a} as a trivial $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module. Let (α, γ) belong to $\mathcal{Z}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})_{\#}$. Now we consider the quadratic extension $\mathfrak{d} := \mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by $(\mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, \theta_{\mathfrak{a}})$ corresponding to $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})_{+}$. Let L_1, L_2 be a basis of \mathfrak{m} and $L_3 := (1/\sqrt{2}) \cdot b_{\mathfrak{m}}(L_1, L_2)$. Furthermore, let $Z_1, Z_2, Z_3 \in \mathfrak{l}^*$ be the dual basis of L_1, L_2, L_3 . Let ω be the 3-form on \mathfrak{g}_{-} that is defined by Equation (7) with respect to the dual basis $\sigma^1, \dots, \sigma^7$ of $Z_1, Z_2, Z_3, A, L_1, L_2, L_3$. It can be checked easily that ω does not depend on the choice of the basis L_1, L_2 of \mathfrak{m} . By definition ω is a nice 3-form.

Proposition 5.3 *Under the above assumptions ω is invariant under \mathfrak{g}_{+} .*

Proof. The computations become simpler when we use the spinorial description of the $G_{2(2)}$ -structure. The orientation defined by ω is such that $Z_1, Z_2, Z_3, A, L_1, L_2, L_3$ is positively oriented. We can define a representation $\phi : \mathcal{C}(\mathfrak{g}_{-}) \rightarrow \text{End}(\Delta_{\mathfrak{g}_{-}})$ of type one by $\phi(Z_i) = \Phi(e_i)$, $\phi(A) = \Phi(e_4)$ and $\phi(L_i) = \Phi(e_{i+4})$, $i = 1, 2, 3$. Then ω corresponds to $\psi = s_1 + s_5$ according to Proposition 2.5. Hence it suffices to show that ψ is \mathfrak{g}_{+} -invariant. From (2) and (4) we obtain

$$\begin{aligned} 4\widetilde{\text{ad}}(Z_{+}) \cdot \psi &= \sum_{i=1}^3 Z_i \cdot [Z_{+}, L_i] \cdot \psi = 2 \sum_{1 \leq i < j \leq 3} Z_{+}([L_i, L_j]) Z_i Z_j \cdot \psi \\ &= 2 \sum_{1 \leq i < j \leq 3} Z_{+}([L_i, L_j]) e_i e_j \cdot (s_1 + s_5) = 4Z_{+}([L_1, L_2]) s_6, \end{aligned} \quad (14)$$

which vanishes by assumption. In the same way we get

$$\begin{aligned} 4\widetilde{\text{ad}}(A_{+}) \cdot \psi &= \sum_{i=1}^3 Z_i \cdot [A_{+}, L_i] \cdot \psi = 2 \sum_{1 \leq i < j \leq 3} \langle A_{+}, \alpha(L_i, L_j) \rangle Z_i Z_j \cdot \psi \\ &= 2 \sum_{1 \leq i < j \leq 3} \langle A_{+}, \alpha(L_i, L_j) \rangle e_i e_j \cdot (s_1 + s_5) = 4\langle A_{+}, \alpha(L_1, L_2) \rangle s_6 = 0. \end{aligned}$$

Furthermore,

$$4\widetilde{\text{ad}}(L_{+}) \cdot \psi = \left(\sum_{i=1}^3 \left(Z_i \cdot [L_{+}, L_i] + L_i \cdot [L_{+}, Z_i] \right) - A \cdot [L_{+}, A] \right) \cdot \psi$$

$$\begin{aligned}
&= \sum_{i=1}^3 \left(Z_i \cdot ([L_+, L_i]_{\mathfrak{l}} + \alpha(L_+, L_i) + \gamma(L_+, L_i, \cdot)) - L_i \cdot Z_i([L_+, \cdot]_{\mathfrak{l}}) \right) \cdot \psi \\
&\quad + A \cdot \langle A, \alpha(L_+, \cdot) \rangle \cdot \psi \\
&= 2 \sum_{i,j=1}^3 Z_j([L_+, L_i]) Z_i L_j \cdot \psi + 2 \sum_{1 \leq i < j \leq 3} \gamma(L_+, L_i, L_j) Z_i Z_j \cdot \psi \\
&\quad - 2 \sum_{k=1}^3 \langle A, \alpha(L_+, L_i) \rangle Z_i A \cdot \psi,
\end{aligned} \tag{15}$$

where we used $\text{tr}(\text{ad} L_+)|_{\mathfrak{l}_-} = 0$. We will show that the right hand side of this equation vanishes. We have

$$\begin{aligned}
&\left(\sum_{i,j=1}^3 Z_j([L_+, L_i]) e_i e_{j+4} + \sum_{i < j} \gamma(L_+, L_i, L_j) e_i e_j - \sum_{k=1}^3 \langle A, \alpha(L_+, L_i) \rangle e_i e_4 \right) \cdot (s_1 + s_5) \\
&= 2 \left(-Z_1([L_+, L_1]) s_5 - Z_1([L_+, L_3]) s_3 - Z_2([L_+, L_2]) s_5 - Z_2([L_+, L_3]) s_8 + \right. \\
&\quad \left. Z_3([L_+, L_1]) s_7 + Z_3([L_+, L_2]) s_4 - Z_3([L_+, L_3]) s_1 \right) + 2\gamma(L_+, L_1, L_2) s_6 + \\
&\quad \sqrt{2} \left(-\langle A, \alpha(L_+, L_1) \rangle s_8 + \langle A, \alpha(L_+, L_2) \rangle s_3 + \langle A, \alpha(L_+, L_3) \rangle s_6 \right).
\end{aligned} \tag{16}$$

Now we use that

$$-Z_1([L_+, L_1]) - Z_2([L_+, L_2]) = -\text{tr ad}(L_+)|_{\mathfrak{m}} = 0,$$

see Remark 4.9. Furthermore,

$$2Z_1([L_+, L_3]) = \sqrt{2} \langle A, \alpha(L_+, L_2) \rangle, \quad 2Z_2([L_+, L_3]) = -\sqrt{2} \langle A, \alpha(L_+, L_1) \rangle$$

since $\sqrt{2}[L_+, L_3] = [L_+, b_{\mathfrak{m}}(L_1, L_2)] = \langle \alpha(L_+, L_2), A \rangle L_1 - \langle \alpha(L_+, L_1), A \rangle L_2$ by (Z2), and

$$Z_3([L_+, L_1]) = Z_3([L_+, L_2]) = Z_3([L_+, L_3]) = 0$$

since $[\mathfrak{l}_+, \mathfrak{l}_-] \subset \mathfrak{m}$. Finally,

$$2\gamma(L_+, L_1, L_2) + \sqrt{2} \langle A, \alpha(L_+, L_3) \rangle = 2\gamma(L_+, L_1, L_2) + \langle A, \alpha(L_+, b_{\mathfrak{m}}(L_1, L_2)) \rangle = 0$$

by (Z3). \square

Corollary 5.4 *If $(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}})$ is a Lie algebra with B-structure, $(\mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, A)$ a pseudo-Euclidean space with distinguished time-like unit vector and if (α, γ) is in $\mathcal{Z}_{\mathbb{Q}}^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})_{\#}$, then $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a}) := (\mathfrak{d}, \theta, \omega, \langle \cdot, \cdot \rangle)$ is a symmetric triple with $G_{2(2)}$ -structure.*

We consider the Lie algebras

$$\begin{aligned}
\mathfrak{h}(1) &= \{[X, Y] = Z\} \\
\mathfrak{g}_{4,1} &= \{B = [L_2, L_3], [B, L_3] = L_1\},
\end{aligned}$$

where we use the following convention. The vectors appearing on the right hand side constitute a basis of the Lie algebra and all brackets of basis vectors not mentioned are equal to zero. The Lie algebra $\mathfrak{h}(1)$ is the three-dimensional Heisenberg algebra, $\mathfrak{g}_{4,1}$ is the only indecomposable real nilpotent Lie algebra of dimension 4.

Proposition 5.5 *Each indecomposable symmetric triple with $G_{2(2)}$ -structure is isomorphic to a quadratic extension $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$ for some $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})_{\#}$, where $(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}})$ is isomorphic to one of the following two triples:*

$$\begin{aligned} \mathfrak{l}_- &= \text{span}\{L_1, L_2, L_3\}, \mathfrak{m} = \text{span}\{L_1, L_2\}, b_{\mathfrak{m}}(L_1, L_2) = \sqrt{2}L_3, \\ \mathfrak{l}_+ &= \text{span}\{B = [L_2, L_3]_{\mathfrak{l}}\}, B \neq 0, \text{ and} \end{aligned}$$

1. $[B, L_1] = [B, L_2]_{\mathfrak{l}} = 0, [B, L_3]_{\mathfrak{l}} = L_1$, or
2. $\text{ad}(B) = 0$.

In the first case \mathfrak{l} is isomorphic to $\mathfrak{g}_{4,1}$, in the second one to $\mathfrak{h}(1) \oplus \mathbb{R}$.

Proof. Let $(\mathfrak{g}, \theta, \omega, \langle \cdot, \cdot \rangle)$ be an indecomposable symmetric triple with $G_{2(2)}$ -structure. Let ω correspond to the pair $(\mathcal{O}, [\psi])$ according to the remark after Definition 3.2. In Section 4.2 we have seen that the intersection \mathfrak{l}_- of the canonical isotropic ideal \mathfrak{i} of \mathfrak{g} with \mathfrak{g}_- is three-dimensional. We proceed as in Section 4.3, especially as in the proof of Proposition 4.6. Let $\varphi \neq 0$ be an isotropic spinor that satisfies $\mathfrak{i}_- \cdot \varphi = 0$. Any choice of a Witt basis \mathbf{b} of \mathfrak{g}_- gives an equivalence $\Delta_{\mathfrak{g}_-} \cong \Delta_{4,3}$. By Proposition 2.1 we can choose \mathbf{b} such that $\psi = s_1 + s_5$ and $\varphi = s_6$. In particular, $\mathfrak{i}_- = \text{span}\{b_1, b_2, b_3\}$, $\mathfrak{i}_-^\perp = \{b_1, b_2, b_3, b_4\}$, and ω is given by (1). We choose an isotropic complement $V_+ \subset \mathfrak{g}_+$ of \mathfrak{i}_+^\perp in \mathfrak{g}_+ , and we put $V_- := \text{span}\{b_5, b_6, b_7\}$ and $V := V_+ \oplus V_-$. Then we may identify V and $\mathfrak{l} := \mathfrak{g}/\mathfrak{i}^\perp$ as vector spaces with involution. Moreover, we put $\mathfrak{a} := (\mathfrak{i} \oplus V)^\perp$ and $A := b_4$. Then \mathfrak{a} is a pseudo-Euclidean space with distinguished time-like unit vector A and the induced involution $\theta_{\mathfrak{a}}$ coincides with $\theta|_{\mathfrak{a}}$. In particular, the symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ is an admissible quadratic extension $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by \mathfrak{a} .

Note that $b(V_-, V_-) \subset V_-$. More exactly,

$$b(b_5, b_6) = \sqrt{2}b_7, \quad b(b_5, b_7) = b(b_6, b_7) = 0.$$

We put $\mathfrak{m} := \text{span}\{b_5, b_6\}$ and $b_{\mathfrak{m}} = b|_{\mathfrak{m} \times \mathfrak{m}}$. We want to show that $(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}})$ is a Lie algebra with B -structure, that the representation ρ of \mathfrak{l} on \mathfrak{a} is trivial and that (α, γ) is in $\mathcal{Z}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$.

The existence of ω implies that \mathfrak{g} is solvable, see Proposition 3.5. Hence, also \mathfrak{l} is solvable. Since $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ is a symmetric triple $[\mathfrak{g}_-, \mathfrak{g}_-] = \mathfrak{g}_+$ holds, which implies $[\mathfrak{l}_-, \mathfrak{l}_-] = \mathfrak{l}_+$. We have to show that ρ is trivial and to verify (L2), (L3) and (Z1) – (Z3). As for ρ , we already know that $\rho|_{\mathfrak{l}_+} = 0$ since \mathfrak{l} is solvable and ρ is semisimple. Even if $\rho \neq 0$ Equation (14) remains true, which implies (L2).

Furthermore,

$$\begin{aligned} 4\widetilde{\text{ad}}(A_+) \cdot \psi &= \sum_{i=1}^3 Z_i \cdot [A_+, L_i] \cdot \psi - A \cdot [A_+, A] \cdot \psi \\ &= \left(2 \sum_{i=1}^3 \langle \rho(L_i)(A_+), A \rangle Z_i A + 2 \sum_{i < j} \langle A_+, \alpha(L_i, L_j) \rangle Z_i Z_j \right) \cdot \psi \\ &= \left(2 \sum_{i=1}^3 \langle \rho(L_i)(A_+), A \rangle e_i e_4 + 2 \sum_{i < j} \langle A_+, \alpha(L_i, L_j) \rangle e_i e_j \right) \cdot (s_1 + s_5) \end{aligned}$$

$$\begin{aligned}
&= 2\sqrt{2}(\langle \rho(L_1)(A_+), A \rangle s_8 - \langle \rho(L_2)(A_+), A \rangle s_3 - \langle \rho(L_3)(A_+), A \rangle s_6) \\
&\quad + 4\langle A_+, \alpha(L_1, L_2) \rangle s_6 = 0
\end{aligned}$$

implies $\rho|_{\mathfrak{m}} = 0$ and

$$\rho(L_3)(A) + \sqrt{2}\alpha(L_1, L_2) = 0. \quad (17)$$

Since $\rho|_{\mathfrak{l}_+} = 0$ Equation (15) remains true. Hence (16) holds and implies (L1), (Z2) and (Z3). It remains to prove $\alpha(\mathfrak{m}, \mathfrak{m}) = 0$ and $\rho|_{\mathfrak{n}} = 0$ and to determine all possible $(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}})$.

Let $(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}})$ be a Lie algebra with B -structure and let (ρ, \mathfrak{a}) be an orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module with $\rho|_{\mathfrak{m}} = 0$. Assume that $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})_+$ is admissible and satisfies (Z2), (Z3) and (17). Let L_1, L_2 be a basis of \mathfrak{m} . Then $B := [L_2, L_3]_{\mathfrak{l}}$ and $C := [L_3, L_1]_{\mathfrak{l}}$ span \mathfrak{l}_+ . The Jacobi identity for L_1, L_2, L_3 implies

$$[B, L_1] + [C, L_2] = 0. \quad (18)$$

Since B and C act nilpotently on \mathfrak{l}_- , we have $\text{tr ad}(B)|_{\mathfrak{m}} = \text{tr ad}(C)|_{\mathfrak{m}} = 0$ by (10). Together with (18) this gives

$$\begin{aligned}
[B, L_1]_{\mathfrak{l}} &= aL_1 + dL_2, & [B, L_2]_{\mathfrak{l}} &= bL_1 - aL_2, \\
[C, L_1]_{\mathfrak{l}} &= dL_1 + cL_2, & [C, L_2]_{\mathfrak{l}} &= -aL_1 - dL_2.
\end{aligned}$$

Because of $[\mathfrak{l}_+, \mathfrak{l}_-] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] = 0$ the Jacobi identity for B, L_2, L_3 and for C, L_1, L_3 gives $[[B, L_2]_{\mathfrak{l}}, L_3]_{\mathfrak{l}} = 0$ and $[[C, L_1]_{\mathfrak{l}}, L_3]_{\mathfrak{l}} = 0$.

First we consider the case that B and C are linearly independent. Then the equation $[[B, L_2]_{\mathfrak{l}}, L_3]_{\mathfrak{l}} = 0$ yields $a = b = 0$ and $[[C, L_1]_{\mathfrak{l}}, L_3]_{\mathfrak{l}} = 0$ gives $d = c = 0$. Hence $\text{ad}(L)|_{\mathfrak{m}} = 0$ for all $L \in \mathfrak{l}_+$. Furthermore, the Jacobi identity for B, L_1, L_3 gives $[B, C] = 0$. Put

$$r := \langle \alpha(B, L_2), A \rangle, \quad s := -\langle \alpha(B, L_1), A \rangle, \quad p := \langle \alpha(C, L_2), A \rangle, \quad q := -\langle \alpha(C, L_1), A \rangle$$

Condition (Z2) yields

$$\sqrt{2}[B, L_3]_{\mathfrak{l}} = rL_1 + sL_2, \quad \sqrt{2}[C, L_3]_{\mathfrak{l}} = pL_1 + qL_2. \quad (19)$$

Assume that $\rho(L_3)(A) = 0$. Then $\alpha(\mathfrak{m}, \mathfrak{m}) = 0$ by (17), thus

$$0 = d\alpha(B, L_1, L_3) = -\alpha([L_1, L_3], B) - \alpha([L_3, B], L_1) - \alpha([B, L_1], L_3) = -\alpha(B, C).$$

In particular, B, C, σ_B, σ_C span an abelian subalgebra of \mathfrak{g}_+ , which is impossible by Proposition 2.1. Hence $\rho(L_3)(A) =: A_0 \neq 0$. Then $\sqrt{2}\alpha(L_1, L_2) = -A_0$. Now

$$\begin{aligned}
d\alpha(B, L_2, L_3) &= \rho(L_3)(\alpha(B, L_2)) - \alpha([L_3, B], L_2) \\
&= -r\rho(L_3)(A) + (1/\sqrt{2})\alpha(rL_1 + sL_2, L_2) \\
&= (-r - r/2)A_0 = 0
\end{aligned}$$

and, analogously,

$$d\alpha(C, L_1, L_3) = (q + q/2)A_0 = 0,$$

which gives $q = r = 0$. Using this we get

$$\begin{aligned} d\alpha(L_1, L_2, L_3) &= \rho(L_3)(\alpha(L_1, L_2)) - \alpha([L_2, L_3], L_1) - \alpha([L_3, L_1], L_2) \\ &= -(1/\sqrt{2})\rho(L_3)(A_0) - \alpha(B, L_1) - \alpha(C, L_2) \\ &= -(1/\sqrt{2})\rho(L_3)(A_0) - sA + pA = 0 \end{aligned} \quad (20)$$

$$\begin{aligned} d\alpha(B, C, L_3) &= \rho(L_3)(\alpha(B, C)) - \alpha([C, L_3], B) - \alpha([L_3, B], C) \\ &= \rho(L_3)(\alpha(B, C)) - (p/\sqrt{2})\alpha(L_1, B) + (s/\sqrt{2})\alpha(L_2, C) \\ &= \rho(L_3)(\alpha(B, C)) + \sqrt{2}psA \end{aligned} \quad (21)$$

$$\begin{aligned} d\alpha(B, L_1, L_3) &= \rho(L_3)(\alpha(B, L_1)) - \alpha([L_1, L_3], B) - \alpha([L_3, B], L_1) \\ &= s\rho(L_3)(A) - \alpha(B, C) + (s/\sqrt{2})\alpha(L_2, L_1) \\ &= -\alpha(B, C) + (s + s/2)A_0 \end{aligned} \quad (22)$$

and, analogously,

$$d\alpha(C, L_2, L_3) = -\alpha(B, C) + (-p - p/2)A_0. \quad (23)$$

Equations (22) and (23) give $s = -p$. Note that $s \neq 0$ since as above $\alpha(B, C) \neq 0$ holds by Prop. 2.1. By (21) and (22) we get $\rho(L_3)(A_0) = (2\sqrt{2}s/3)A$. On the other hand, $\rho(L_3)(A_0) = -2\sqrt{2}sA$ by (20), which gives a contradiction.

Now suppose $\dim \mathfrak{l}_+ = 1$. We may assume $[L_3, L_1]_{\mathfrak{l}} = 0$ and $\mathfrak{l}_+ = \mathbb{R} \cdot B$ with $B := [L_2, L_3]_{\mathfrak{l}}$. The Jacobi identity for L_1, L_2, L_3 gives $[B, L_1]_{\mathfrak{l}} = 0$. Recall from Remark 4.9 that $\text{tr ad}(B)|_{\mathfrak{m}} = 0$, thus $[B, L_2]_{\mathfrak{l}} = bL_1$ for some $b \in \mathbb{R}$. Put

$$r := \langle A, \alpha(B, L_2) \rangle, \quad s := -\langle A, \alpha(B, L_1) \rangle.$$

Then Condition (Z2) gives

$$\sqrt{2}[B, L_3] = rL_1 + sL_2.$$

As above, $A_0 := \rho(L_3)(A)$. Computations analogous to (20) and (22) give

$$\begin{aligned} d\alpha(L_1, L_2, L_3) &= -(1/\sqrt{2})\rho(L_3)(A_0) - sA = 0 \\ d\alpha(B, L_1, L_3) &= (s + s/2)A_0 = 0, \end{aligned}$$

which implies $A_0 = 0$ and $s = 0$. In particular, $\alpha(B, L_1) = 0$ and $\sqrt{2}[B, L_3] = rL_1$.

From $d\alpha = 0$ we get $\alpha([B, L_2], L_3) = \alpha([B, L_3], L_2) = (r/\sqrt{2})\alpha(L_1, L_2) = 0$, thus $b\alpha(L_1, L_3) = 0$. For $b \neq 0$ we would get $\alpha(L_1, \cdot) = 0$. On the other hand, in this case $\mathfrak{l} \cong \mathfrak{g}_{4,1}$ and for $\mathfrak{g}_{4,1}$ the admissibility condition (A_2) in [KO2], Definition 5.3. implies that a cocycle (α, γ) can only be admissible if $\alpha(\mathfrak{z}, \cdot) = \alpha(L_1, \cdot) \neq 0$, see also [K2], Prop. 4.4 for this fact. Hence $b = 0$. Rescaling L_1, L_2 and L_3 we may assume $r \in \{0, 1\}$.

If $B = C = 0$, then the Killing form of \mathfrak{l} vanishes, thus $\rho = 0$ by Proposition 4.1. Since $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ is indecomposable this yields $\mathfrak{a}_- = \alpha(\mathfrak{l}_+, \mathfrak{l}_-) = 0$, which contradicts $\dim \mathfrak{a}_- = 1$. \square

6 Classification

6.1 Classification in terms of $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$

In this subsection we want to describe the isomorphism classes of quadratic extensions $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$ in terms of $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$.

For a Lie algebra with B -structure $(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}})$ we define

$$\mathcal{G} := \left\{ S \mid \begin{array}{l} S \in \text{Aut}(\mathfrak{l}, \theta_{\mathfrak{l}}), \ S(\mathfrak{m}) = \mathfrak{m}, \\ \text{pr}_{\mathfrak{n}}(S(b_{\mathfrak{m}}(L, L'))) = b_{\mathfrak{m}}(S(L), S(L')) \text{ for all } L, L' \in \mathfrak{m} \end{array} \right\}.$$

Then \mathcal{G} equals the semi-direct product $\text{Aut}(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}) \ltimes \mathcal{N}$, where \mathcal{N} was defined in (13) and

$$\text{Aut}(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}) = \{S \in \mathcal{G} \mid S(\mathfrak{n}) = \mathfrak{n}\}$$

is the automorphism group of $(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}})$.

Definition 6.1 *We will say that $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$ and $\mathfrak{d}_{\alpha', \gamma'}(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$ are equivalent if there is an isomorphism $\Psi_0 : \mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a}) \rightarrow \mathfrak{d}_{\alpha', \gamma'}(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$ of symmetric triples with $G_{2(2)}$ -structure such that*

$$(i) \ \Psi_0(\mathfrak{l}^*) = \mathfrak{l}^* \text{ (hence } \Psi_0(\mathfrak{l}^* \oplus \mathfrak{a}) = \mathfrak{l}^* \oplus \mathfrak{a}),$$

$$(ii) \ \Psi_0|_{\mathfrak{a}} = \text{id} \mod \mathfrak{l}^*,$$

$$(iii) \text{ the map } S_0 := \text{pr}_{\mathfrak{l}} \circ \Psi_0|_{\mathfrak{l}} : \mathfrak{l} \rightarrow \mathfrak{l} \text{ belongs to } \mathcal{N}.$$

Proposition 6.2 *The quadratic extensions $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$ and $\mathfrak{d}_{\alpha', \gamma'}(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$ are equivalent if and only if $[\alpha, \gamma] = [\alpha', \gamma'] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$.*

Proof. In [KO2] we proved that the isomorphisms $\Psi_0 : \mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}) \rightarrow \mathfrak{d}_{\alpha', \gamma'}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ of symmetric triples that satisfy Properties (i) – (iii) of Definition 6.1 are exactly the linear maps

$$\Psi_0 = \begin{pmatrix} (S_0^*)^{-1} & -(S_0^*)^{-1}\tau^* & (S_0^*)^{-1}(\bar{\sigma} - \frac{1}{2}\tau^*\tau) \\ 0 & \text{id} & \tau \\ 0 & 0 & S_0 \end{pmatrix} : \mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l} \longrightarrow \mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l}, \quad (24)$$

where (τ, σ) is in $\mathcal{C}_Q^1(\mathfrak{l}, \mathfrak{a})_+$ for $\sigma(L_1, L_2) := \bar{\sigma}(L_1)(L_2)$, $S_0 : \mathfrak{l} \rightarrow \mathfrak{l}$ is in $\text{Aut}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{m})$ and $(\alpha, \gamma) = (S_0^*\alpha', S_0^*\gamma') \cdot (\tau, \sigma)$. It remains to decide which of these maps preserve the $G_{2(2)}$ -structure. Choose a basis L_1, L_2 of \mathfrak{m} and put $L_3 := \sqrt{2}b_{\mathfrak{m}}(L_1, L_2)$. Hence L_1, L_2, L_3 is a basis of \mathfrak{l} . Denote by Z_1, Z_2, Z_3 the dual basis of \mathfrak{l}^* . With respect to these bases we have

$$S_0(L_3) = L_3 + s_1L_1 + s_2L_2, \quad \tau(L_i) = t_iA, \quad i = 1, 2, 3.$$

Since

$$\begin{aligned} b(\Psi_0(Z_3), \Psi_0(L_3)) &= b(Z_3, L_3 + s_1L_1 + s_2L_2 + t_3A) \\ &= -A + s_1\sqrt{2}Z_2 - s_2\sqrt{2}Z_1 - t_3Z_3 \end{aligned}$$

and

$$\Psi_0(b(Z_3, L_3)) = \Psi_0(-A) = -A - t_1 Z_1 - t_2 Z_2 - (t_3 + s_1 t_1 + s_2 t_2) Z_3$$

the equation $b(\Psi_0(Z_3), \Psi_0(L_3)) = \Psi_0(b(Z_3, L_3))$ is equivalent to

$$t_1 = \sqrt{2}s_2, \quad t_2 = -\sqrt{2}s_1,$$

hence to (B1). Furthermore,

$$\begin{aligned} b(\Psi_0(Z_1), \Psi_0(L_1)) &= b(Z_1 - s_1 Z_3, L_1 + t_1 A + (\sigma(L_1, L_2) + \frac{1}{2} t_1 t_2) Z_2) \\ &= A + t_1 Z_1 - s_1 \sqrt{2} Z_2 + \sqrt{2} \sigma(L_1, L_2) Z_3 \end{aligned}$$

and

$$\Psi_0(b(Z_1, L_1)) = \Psi_0(A) = A + t_1 Z_1 + t_2 Z_2 + (t_3 - s_1 t_1 - s_2 t_2) Z_3.$$

Hence, if (B1) holds, then $b(\Psi_0(Z_1), \Psi_0(L_1)) = \Psi_0(b(Z_1, L_1))$ is equivalent to (B2). Moreover, a direct calculation shows that (B1) and (B2) imply $b(X, Y) = \Psi_0(b(X, Y))$ for all other combinations of basis vectors X and Y . \square

Proposition 6.3 *Let $(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}})$ and $(\mathfrak{l}', \theta_{\mathfrak{l}'}, b_{\mathfrak{m}'})$ be Lie algebras with B -structure. Let \mathfrak{a} and \mathfrak{a}' be pseudo-Euclidean spaces with distinguished time-like unit vectors A and A' , respectively. Furthermore, let $(\alpha, \gamma), (\alpha', \gamma') \in \mathcal{Z}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$ be admissible. Then $\mathfrak{d} := \mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$ and $\mathfrak{d}' := \mathfrak{d}_{\alpha', \gamma'}(\mathfrak{l}', \theta_{\mathfrak{l}'}, b_{\mathfrak{m}'}, \mathfrak{a}')$ are isomorphic as symmetric triples with $G_{2(2)}$ -structure if and only if there are an isomorphism $S : (\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}) \rightarrow (\mathfrak{l}', \theta_{\mathfrak{l}'}, b_{\mathfrak{m}'})$ and an isometry $U : \mathfrak{a}' \rightarrow \mathfrak{a}$ satisfying $U(A') = A$ such that $(S, U)^*[\alpha', \gamma'] = [\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$.*

Remark 6.4 In the above proposition the map

$$(S, U)^* : \mathcal{H}_Q^2(\mathfrak{l}', \theta_{\mathfrak{l}'}, b_{\mathfrak{m}'}, \mathfrak{a}') \rightarrow \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$$

is well defined. Indeed, it is easy to check, that $(S, U)^*$ maps $\mathcal{Z}_Q^2(\mathfrak{l}', \theta_{\mathfrak{l}'}, b_{\mathfrak{m}'}, \mathfrak{a}')$ to $\mathcal{Z}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$. Moreover, assume that (α'_1, γ'_1) and (α'_2, γ'_2) are equivalent elements of $\mathcal{Z}_Q^2(\mathfrak{l}', \theta_{\mathfrak{l}'}, b_{\mathfrak{m}'}, \mathfrak{a}')$. Let $\mathcal{N} \subset \text{Aut}(\mathfrak{l}, \theta_{\mathfrak{l}})$ and $\mathcal{N}' \subset \text{Aut}(\mathfrak{l}', \theta_{\mathfrak{l}'})$ be the subgroups defined by (13). Then $(\alpha'_1, \gamma'_1) = ((S'_0)^* \alpha'_2, (S'_0)^* \gamma'_2)(\tau, \sigma)$ for some $S'_0 \in \mathcal{N}'$ and $(\tau, \sigma) \in \mathcal{C}_Q^1(\mathfrak{l}', \mathfrak{a}')_+$ that satisfy Properties (B1) and (B2). If we put $S_0 := S^{-1} S'_0 S$, then S_0 is in \mathcal{N} and

$$(S, U)^*(\alpha'_1, \gamma'_1) = (S_0^* (S, U)^*(\alpha'_2, \gamma'_2)) \cdot (S, U)^*(\tau, \sigma).$$

Hence it suffices to show that $S_0 \in \mathcal{N}$ and $(S, U)^*(\tau, \sigma) \in \mathcal{C}_Q^1(\mathfrak{l}, \mathfrak{a})_+$ satisfy (B1) and (B2). As for (B1), one checks first that $\text{tr}(\text{pr}_{\mathfrak{m}} S_0(b_{\mathfrak{m}}(L, \cdot))) = \text{tr}(\text{pr}_{\mathfrak{m}'} S'_0(b_{\mathfrak{m}'}(SL, \cdot)))$, then (B1) follows easily. Condition (B2) is easy to check.

Proof of Prop. 6.3. Let us first assume that there exist an isomorphism $S : (\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}) \rightarrow (\mathfrak{l}', \theta_{\mathfrak{l}'}, b_{\mathfrak{m}'})$ and an isometry $U : \mathfrak{a}' \rightarrow \mathfrak{a}$ satisfying $U(A') = A$ such that $(S, U)^*[\alpha', \gamma'] = [\alpha, \gamma]$. Then $\mathfrak{d}_{(S, U)^*(\alpha', \gamma')}(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$ and \mathfrak{d} are equivalent, thus isomorphic. Furthermore, it can be easily checked that

$$\Psi_1 : \mathfrak{d}_{(S, U)^*(\alpha', \gamma')}(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a}) \rightarrow \mathfrak{d}', \quad \Psi_1(Z + A + L) = (S^*)^{-1}(Z) + U^{-1}(A) + S(L)$$

for $Z \in \mathfrak{l}^*$, $A \in \mathfrak{a}$, $L \in \mathfrak{l}$ is an isomorphism.

Now let $\Psi : \mathfrak{d} \rightarrow \mathfrak{d}'$ be an isomorphism. In particular, Ψ is an isomorphism of symmetric triples. Since (α, γ) and (α', γ') are admissible, we can apply Proposition 6.1. in [KO2]. Hence there is an isomorphism of triples $(\tilde{S}, U)(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}) \rightarrow (\mathfrak{l}', \theta_{\mathfrak{l}'}, \mathfrak{a}')$ such that $(\tilde{S}, U)^*(\alpha', \gamma') = (\alpha, \gamma)(\sigma, \tau)$ for some $(\sigma, \tau) \in \mathcal{C}_Q^1(\mathfrak{l}, \mathfrak{a})_+$. The maps $\tilde{S} : \mathfrak{l} \rightarrow \mathfrak{l}'$ and $U^{-1} : \mathfrak{a} \rightarrow \mathfrak{a}'$ are induced by Ψ , which maps the canonical isotropic ideal $\mathfrak{i} = \mathfrak{l}^*$ of \mathfrak{d} to the canonical isotropic ideal $\mathfrak{i}' = (\mathfrak{l}')^*$ and is therefore compatible with the filtrations $\mathfrak{l}^* \subset \mathfrak{l}^* \oplus \mathfrak{a} \subset \mathfrak{d}$ and $(\mathfrak{l}')^* \subset (\mathfrak{l}')^* \oplus \mathfrak{a}' \subset \mathfrak{d}'$. Furthermore, $\Psi|_{\mathfrak{l}^*} = (\tilde{S}^*)^{-1} : \mathfrak{l}^* \rightarrow (\mathfrak{l}')^*$.

As above, $\mathfrak{l} = \mathfrak{m} \oplus \mathfrak{n}$, $\mathfrak{l}' = \mathfrak{m}' \oplus \mathfrak{n}'$. Then $(\tilde{S}^*)^{-1}(\mathfrak{n}^*) = (\mathfrak{n}')^*$ since Ψ must map $b(\mathfrak{i}, \mathfrak{i})$ to $b'(\mathfrak{i}', \mathfrak{i}')$. Consequently, $\tilde{S}(\mathfrak{m}) = \mathfrak{m}'$. Moreover,

$$\begin{aligned} \text{pr}_{\mathfrak{n}'} \tilde{S}(b_{\mathfrak{m}}(L, L')) &= \text{pr}_{\mathfrak{n}'} \Psi(b(L, L')) = \text{pr}_{\mathfrak{n}'}(b'(\Psi(L), \Psi(L'))) \\ &= b'(\text{pr}_{\mathfrak{m}'} \Psi(L), \text{pr}_{\mathfrak{m}'} \Psi(L')) = b_{\mathfrak{m}'}(\tilde{S}(L), \tilde{S}(L')) \end{aligned} \quad (25)$$

for all $L, L' \in \mathfrak{m}$. Define $S : (\mathfrak{l}, \theta_{\mathfrak{l}}) \rightarrow (\mathfrak{l}', \theta_{\mathfrak{l}'})$ and $S_0 : (\mathfrak{l}, \theta_{\mathfrak{l}}) \rightarrow (\mathfrak{l}, \theta_{\mathfrak{l}})$ by $\tilde{S} = S \circ S_0$ and

$$S(\mathfrak{m}) = \mathfrak{m}', \quad S(\mathfrak{n}) = \mathfrak{n}', \quad S_0|_{\mathfrak{m}} = \text{id}_{\mathfrak{m}}, \quad \text{pr}_{\mathfrak{n}} S_0|_{\mathfrak{n}} = \text{id}_{\mathfrak{n}}, \quad S_0|_{\mathfrak{l}_+} = \text{id}_{\mathfrak{l}_+}.$$

Then S_0 is an automorphism of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by Proposition 5.5. Hence S is also an automorphism of $(\mathfrak{l}, \theta_{\mathfrak{l}})$. Thus $S_0 \in \mathcal{N}$ and, by (25), $S : (\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}) \rightarrow (\mathfrak{l}', \theta_{\mathfrak{l}'}, b_{\mathfrak{m}'})$ is an isomorphism. Since the $G_{2(2)}$ -structures on \mathfrak{d} and \mathfrak{d} define orientations on $\mathfrak{d}_-/\mathfrak{l}_-^*$ and $\mathfrak{d}'_-(\mathfrak{l}')_-^*$, respectively, we obtain $U(A') = A$. Moreover,

$$(\alpha, \gamma)(\sigma, \tau) = (\tilde{S}, U)^*(\alpha', \gamma') = (S_0, \text{id})^*(S, U)^*(\alpha', \gamma'),$$

which proves the claim. \square

6.2 Computation of $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})_0$

In the following we will consider the subset $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})_0 \subset \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$ of those elements $[\alpha, \beta] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$ for which (α, γ) is admissible and indecomposable as an element of $\mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})_+$. This set is acted upon by the group

$$G := \text{Aut}(\mathfrak{l}, \theta, b_{\mathfrak{m}}) \times \text{Aut}(\mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, A),$$

where $\text{Aut}(\mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, A) = \{U \in \text{O}(\mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}), U(A) = A\}$.

According to Propositions 5.5 and 6.3, in order to give a classification of symmetric triples with $G_{2(2)}$ -structure it remains to compute $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})_0/G$ for $\mathfrak{l} = \mathfrak{g}_{4,1}$ and $\mathfrak{l} = \mathbb{R} \oplus \mathfrak{h}(1)$ with $\theta_{\mathfrak{l}}$ and $b_{\mathfrak{m}}$ as given in Proposition 5.5.

For the moment let us forget about the additional structures $\theta_{\mathfrak{l}}$ and $b_{\mathfrak{m}}$ and consider $\mathfrak{g}_{4,1}$ and $\mathbb{R} \oplus \mathfrak{h}(1)$ just as Lie algebras. In [KO1] we introduced the quadratic cohomology set $\mathcal{H}_Q^2(\mathfrak{l}, \mathfrak{a})$ for a Lie algebra \mathfrak{l} and an arbitrary orthogonal \mathfrak{l} -module \mathfrak{a} . In [K2] we determined this cohomology for $\mathfrak{l} \in \{\mathfrak{g}_{4,1}, \mathbb{R} \oplus \mathfrak{h}(1)\}$ and trivial \mathfrak{l} -modules \mathfrak{a} . Let us recall some intermediate results of this computation, which will be useful in the following. Let $\mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})$ be the set of quadratic cocycles of a Lie algebra \mathfrak{l} with coefficients in the pseudo-Euclidean vector space \mathfrak{a} considered as a trivial \mathfrak{l} -module. If a cocycle

$(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})_+$ is admissible, then it is balanced as an element of $\mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})$ in the sense of [K2]. Moreover, if $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})$ is indecomposable and contained in $\mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})_+$, then it is indecomposable as an element of $\mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})_+$.

In the following $Z_1, Z_2, Z_3, Z_B \in \mathfrak{l}^*$ will denote the dual basis to $L_1, L_2, L_3, B \in \mathfrak{l}$.

Proposition 6.5 1. Let \mathfrak{l} be the Lie algebra $\mathfrak{g}_{4,1} = \{[L_2, L_3] = B, [B, L_3] = L_1\}$.

- (a) If $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})$ is balanced, then $\alpha(L_1, \cdot) \neq 0$.
(b) If $\mathfrak{a} \in \{\mathbb{R}^{1,1}, \mathbb{R}^{2,0}\}$ and A, A_1 is an orthonormal basis of \mathfrak{a} , then the cocycle $(\alpha_1, s\gamma_1) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})$ with

$$\alpha_1 = (Z_1 \wedge Z_3) \otimes A_1 + (Z_2 \wedge Z_B) \otimes A, \quad \gamma_1 = Z_B \wedge Z_1 \wedge Z_3 \quad (26)$$

is balanced and indecomposable for all $s \in \mathbb{R}$.

2. Let \mathfrak{l} be the Lie algebra $\mathbb{R} \oplus \mathfrak{h}(1) = \mathbb{R} \cdot L_1 \oplus \{[L_2, L_3] = B\}$.

If $\mathfrak{a} \in \{\mathbb{R}^{1,1}, \mathbb{R}^{2,0}\}$ and A, A_1 is an orthonormal basis of \mathfrak{a} , then $(\alpha_2, \gamma_2) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})$ with

$$\alpha_2 = (Z_1 \wedge Z_3) \otimes A_1 + (Z_B \wedge Z_3) \otimes A, \quad \gamma_2 = Z_B \wedge Z_1 \wedge Z_2 \quad (27)$$

is balanced and indecomposable. If $\mathfrak{a} = \mathbb{R}^{1,0}$, then $(\alpha_3, \gamma_3) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})$ with

$$\alpha_3 = (Z_B \wedge Z_3) \otimes A, \quad \gamma_3 = Z_B \wedge Z_1 \wedge Z_2 \quad (28)$$

is balanced and indecomposable.

Proof. [K2], Prop. 4.4 for $\mathfrak{g}_{4,1}$ and Prop. 4.5 for $\mathfrak{h}(1) \oplus \mathbb{R}$. □

Proposition 6.6 If $(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}})$ is defined as in item 1 of Prop. 5.5, then the orbit space $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})_0/G$ is not empty if and only if \mathfrak{a} is isomorphic to $\mathbb{R}^{1,1}$ or $\mathbb{R}^{2,0}$. For $\mathfrak{a} \in \{\mathbb{R}^{1,1}, \mathbb{R}^{2,0}\}$ the elements of $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})_0/G$ are represented by $[\alpha, \gamma_t]$, $t \in \mathbb{R}$, with

$$\begin{aligned} \alpha &= (-\sqrt{2}Z_B \wedge Z_2) \otimes A + (Z_1 \wedge Z_3) \otimes A_1, \\ \gamma_t &= tZ_B \wedge Z_1 \wedge Z_3, \end{aligned}$$

where A_1 is a fixed unit vector in \mathfrak{a}_+ .

Proof. Let $[\alpha, \gamma]$ be in $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})_0$. Then $\alpha(L_1, L_2) = 0$. Condition (Z2) gives

$$[B, \sqrt{2}L_3] = \langle A, \alpha(B, L_2) \rangle L_1 - \langle A, \alpha(B, L_1) \rangle L_2 = \sqrt{2}L_1,$$

hence $\alpha(B, L_2) = -\sqrt{2}A$ and $\alpha(B, L_1) = 0$. Because of admissibility $\alpha(L_1, L_3)$ cannot vanish, see Prop. 6.5, 1. (a). Moreover, $\alpha(L_1, L_3) \neq 0$ must span \mathfrak{a}_+ because of indecomposability. Let $S_0 \in \mathcal{N}$ be given by $S_0(L_3) = s_1L_1 + s_2L_2 + L_3$. Then

$$(S_0^* \alpha + d\tau) = \alpha - (Z_2 \wedge Z_3) \otimes \tau(B) - 2\sqrt{2}s_2(Z_B \wedge Z_3) \otimes A \quad (29)$$

since $\tau(L_1) = \sqrt{2}s_2A$ by (B1). Hence we may assume $\alpha(L_2, L_3) = \alpha(B, L_3) = 0$, thus

$$\alpha = (-\sqrt{2}Z_B \wedge Z_2) \otimes A + (sZ_1 \wedge Z_3) \otimes A_1$$

for some unit vector $A_1 \in \mathfrak{a}_+$ and some $s \in \mathbb{R}$, $s \neq 0$. It is easy to check that each such α satisfies $d\alpha = 0$. Moreover, for different choices of s the quadratic cocycles $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})$ are not equivalent. Obviously, $\langle \alpha \wedge \alpha \rangle = 0$ and, moreover, $d\gamma = 0$ for each $\theta_{\mathfrak{l}}$ -invariant $\gamma \in C^3(\mathfrak{l})$.

As for γ , note that

$$2\gamma(B, L_1, L_2) = -\langle A, \alpha(B, \sqrt{2}L_3) \rangle = 0 \quad (30)$$

because of Condition (Z3).

Let us now check how we can change γ without changing α and $[\alpha, \gamma]$. By (29), $(S_0^*\alpha + d\tau) = \alpha$ holds if and only if $\tau(B) = 0$ and $s_2 = 0$, which is equivalent to $d\tau = 0$. Let τ and S_0 satisfy these conditions. Then (30) implies $S_0^*\gamma = \gamma$. Hence

$$S_0^*\gamma + d\sigma + \langle (S_0^*\alpha + \frac{1}{2}d\tau) \wedge \tau \rangle = \gamma + d\sigma + \langle \alpha \wedge \tau \rangle. \quad (31)$$

Since

$$d\sigma = \sigma(L_1, L_2) \cdot Z_B \wedge Z_2 \wedge Z_3 = -(1/\sqrt{2})\langle \tau(L_3), A \rangle \cdot Z_B \wedge Z_2 \wedge Z_3$$

by Condition (B2) and

$$\langle \alpha \wedge \tau \rangle = \langle \alpha(B, L_2), \tau(L_3) \rangle \cdot Z_B \wedge Z_2 \wedge Z_3 = \langle -\sqrt{2}A, \tau(L_3) \rangle \cdot Z_B \wedge Z_2 \wedge Z_3$$

the right hand side of (31) equals

$$\gamma - (3/\sqrt{2})\langle \tau(L_3), A \rangle \cdot Z_B \wedge Z_2 \wedge Z_3.$$

Hence we may assume $\gamma(B, L_2, L_3) = 0$. Together with (30) this implies

$$\gamma = tZ_B \wedge Z_1 \wedge Z_3$$

for some $t \in \mathbb{R}$. It remains to decide which of these (α, γ) are admissible and indecomposable and to divide by G .

It holds that $S \in \text{Aut}(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}})$ and $U \in \text{Aut}(\mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, A)$ if and only if

$$S(L_1) = a^3L_1, \quad S(L_2) = bL_1 + (1/a) \cdot L_2, \quad S(L_3) = a^2L_3, \quad S(B) = aB,$$

$$U(A) = A, \quad U(A_1) = \delta A_1, \quad \delta = \pm 1,$$

for some $a, b \in \mathbb{R}$, $a \neq 0$. Hence $[\alpha, \gamma]$ is in the same orbit as $[\alpha', \gamma']$ with

$$\begin{aligned} \alpha' &= (-\sqrt{2}Z_B \wedge Z_2) \otimes A + (Z_1 \wedge Z_3) \otimes A_1, \\ \gamma' &= t'Z_B \wedge Z_1 \wedge Z_3 \end{aligned}$$

for some $t' \in \mathbb{R}$. For different choices of the parameters t' the equivalence classes $[\alpha', \gamma']$ belong to different G -orbits. As a cohomology class in $\mathcal{H}_Q^2(\mathfrak{l}, \mathfrak{a})$, $[\alpha', \gamma']$ equals

$$[(\sqrt{2}Z_2 \wedge Z_B) \otimes A + (Z_1 \wedge Z_3) \otimes A_1, t'Z_B \wedge Z_1 \wedge Z_3],$$

which is in the same $\text{Aut}(\mathfrak{l}) \times \text{O}(\mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}})$ -orbit as $[\alpha_1, s\gamma_1]$ for some $s \in \mathbb{R}$ with α_1 and γ_1 as in (26). Hence (α', γ') admissible and indecomposable. \square

Proposition 6.7 *If $(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}})$ is defined as in item 2 of Prop. 5.5, then the orbit space $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})_0/G$ is not empty if and only if \mathfrak{a} is isomorphic to $\mathbb{R}^{1,1}$, $\mathbb{R}^{2,0}$ or $\mathbb{R}^{1,0}$. Put*

$$\gamma_0 = (1/\sqrt{2}) \cdot Z_B \wedge Z_1 \wedge Z_2. \quad (32)$$

For $\mathfrak{a} \in \{\mathbb{R}^{1,1}, \mathbb{R}^{2,0}\}$ we have $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})_0/G = \{[\bar{\alpha}, \gamma_0]\}$ with

$$\bar{\alpha} = (Z_B \wedge Z_3) \otimes A + (Z_1 \wedge Z_3) \otimes A_1, \quad (33)$$

where A_1 is a fixed unit vector in \mathfrak{a}_+ .

For $\mathfrak{a} = \mathbb{R}^{1,0}$ we have $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})_0/G = \{[\alpha_0, \gamma_0]\}$, where

$$\alpha_0 = (Z_B \wedge Z_3) \otimes A.$$

Proof. Let $[\alpha, \gamma]$ be in $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}}, \mathfrak{a})_0$. Then $\alpha(L_1, L_2) = 0$. Condition (Z2) gives

$$[B, \sqrt{2}L_3] = \langle A, \alpha(B, L_2) \rangle L_1 - \langle A, \alpha(B, L_1) \rangle L_2 = 0,$$

hence $\alpha(B, L_1) = \alpha(B, L_2) = 0$. Let $S_0 \in \mathcal{N}$ be given by $S_0(L_3) = s_1 L_1 + s_2 L_2 + L_3$. Then

$$(S_0^* \alpha + d\tau) = \alpha - (Z_2 \wedge Z_3) \otimes \tau(B). \quad (34)$$

Thus we may assume $\alpha(L_2, L_3) = 0$. Moreover, $\alpha(L_1, L_3)$ must span \mathfrak{a}_+ . Thus we have

$$\alpha = (tZ_B \wedge Z_3) \otimes A + (t'Z_1 \wedge Z_3) \otimes A_1$$

for some $t' \neq 0$ and a unit vector $A_1 \in \mathfrak{a}_+$, if $\mathfrak{a}_+ \neq 0$ and

$$\alpha = (tZ_B \wedge Z_3) \otimes A$$

if $\mathfrak{a}_+ = 0$. In both cases $t \neq 0$ because of indecomposability. Obviously, each such α satisfies $d\alpha = 0$ and $\langle \alpha \wedge \alpha \rangle = 0$. For different choices of t and t' the quadratic cocycles are in different equivalence classes. Condition (Z3) gives

$$2\gamma(B, L_1, L_2) = -\langle A, \alpha(B, \sqrt{2}L_3) \rangle = -\langle A, \sqrt{2}tA \rangle = \sqrt{2}t.$$

This is the only condition for γ since $d\gamma = 0$ for each $\theta_{\mathfrak{l}}$ -invariant $\gamma \in C^3(\mathfrak{l})$. Let us now check how we can change γ without changing α and $[\alpha, \gamma]$. By (34) $(S_0^* \alpha + d\tau) = \alpha$ holds if and only if $\tau(B) = 0$. Moreover, (B1) implies $\tau(L_1) = \sqrt{2}s_2$ and $\tau(L_2) = -\sqrt{2}s_1$. Let τ and S_0 satisfy these conditions. Then

$$S_0^* \gamma + d\sigma + \langle (S_0^* \alpha + \frac{1}{2}d\tau) \wedge \tau \rangle = S_0^* \gamma + \langle \alpha \wedge \tau \rangle. \quad (35)$$

Since

$$S_0^* \gamma = \gamma + (s_2 t / \sqrt{2}) \cdot Z_B \wedge Z_1 \wedge Z_3 - (s_1 t / \sqrt{2}) \cdot Z_B \wedge Z_2 \wedge Z_3$$

and

$$\langle \alpha \wedge \tau \rangle = \sqrt{2}s_2 t Z_B \wedge Z_1 \wedge Z_3 - \sqrt{2}s_1 t Z_B \wedge Z_2 \wedge Z_3.$$

Equation (35) implies that we may assume $\gamma = (t/\sqrt{2}) \cdot Z_B \wedge Z_1 \wedge Z_2$.

It holds that $S \in \text{Aut}(\mathfrak{l}, \theta_{\mathfrak{l}}, b_{\mathfrak{m}})$ if and only if

$$S(L_1) = aL_1, \quad S(L_2) = bL_2 + xL_1, \quad S(L_3) = abL_3, \quad S(B) = ab^2B$$

and $U \in \text{Aut}(\mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, \theta_{\mathfrak{a}}, A)$ if and only $U = \text{id}$ in case $\mathfrak{a} = \mathbb{R}^{1,0}$ and

$$U(A) = A, \quad U(A_1) = \delta A_1, \quad \delta = \pm 1$$

if $\mathfrak{a} \in \{\mathbb{R}^{2,0}, \mathbb{R}^{1,1}\}$. Hence, if $\mathfrak{a} \in \{\mathbb{R}^{2,0}, \mathbb{R}^{1,1}\}$, then $[\alpha, \gamma]$ is in the same orbit as $[\bar{\alpha}, \gamma_0]$ with $\bar{\alpha}$ and γ_0 as defined in (33) and (32). As a cohomology class in $\mathcal{H}_Q^2(\mathfrak{l}, \mathfrak{a})$, $[\bar{\alpha}, \gamma_0]$ is in the same $\text{Aut}(\mathfrak{l}) \times \text{O}(\mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}})$ -orbit as $[\alpha_2, \gamma_2]$ with α_2 and γ_2 as in (27). Hence $(\bar{\alpha}, \gamma_0)$ is admissible and indecomposable.

If $\mathfrak{a} = \mathbb{R}^{1,0}$, then $[\alpha, \gamma]$ is in the same G -orbit as $[\alpha_0, \gamma_0]$. As a cohomology class in $\mathcal{H}_Q^2(\mathfrak{l}, \mathfrak{a})$, $[\alpha_0, \gamma_0]$ is in the same $\text{Aut}(\mathfrak{l}) \times \text{O}(\mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}})$ -orbit as $[\alpha_3, \gamma_3]$ for α_3, γ_3 as in (28). Hence (α_0, γ_0) is admissible and indecomposable. \square

6.3 Classification result

Recall that for a given Lie algebra \mathfrak{l} with involution $\theta_{\mathfrak{l}}$, a pseudo-Euclidean space $(\mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}})$ with isometric involution $\theta_{\mathfrak{a}}$, a suitable 2-cocycle $\alpha \in Z^2(\mathfrak{l}, \mathfrak{a})$ and a suitable 3-form $\gamma \in \wedge^3 \mathfrak{l}^*$, we can define a symmetric triple $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}) = (\mathfrak{d}, \theta, \langle \cdot, \cdot \rangle)$ by $\mathfrak{d} = \mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l}$ (as a vector space), $\theta = \theta_{\mathfrak{l}}^* \oplus \theta_{\mathfrak{a}} \oplus \theta_{\mathfrak{l}}$, $[\mathfrak{l}^* \oplus \mathfrak{a}, \mathfrak{l}^* \oplus \mathfrak{a}] = 0$,

$$\begin{aligned} [L, L'] &= \gamma(L, L', \cdot) + \alpha(L, L') + [L, L']_{\mathfrak{l}} \\ [L, A + Z] &= -\langle A, \alpha(L, \cdot) \rangle + \text{ad}^*(L)(Z) \end{aligned}$$

and

$$\langle Z + A + L, Z' + A' + L' \rangle = \langle A, A' \rangle_{\mathfrak{a}} + Z(L') + Z'(L)$$

for $Z, Z' \in \mathfrak{l}^*$, $A, A' \in \mathfrak{a}$ and $L, L' \in \mathfrak{l}$.

In the following theorem $Z_1, Z_2, Z_3, Z_B \in \mathfrak{l}^*$ is the dual basis to $L_1, L_2, L_3, B \in \mathfrak{l}$.

Theorem 6.8 *If $(\mathfrak{g}, \theta, \omega, \langle \cdot, \cdot \rangle)$ is an indecomposable symmetric triple with $G_{2(2)}$ -structure, then it is isomorphic to exactly one of the symmetric triples with $G_{2(2)}$ -structure $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}), \omega)$ for the following data $\mathfrak{l}, \mathfrak{a}, \alpha, \gamma$ and ω :*

1. $\mathfrak{l} = \mathfrak{g}_{4,1} = \{B = [L_2, L_3]_{\mathfrak{l}}, [B, L_3]_{\mathfrak{l}} = L_1\}$, $\mathfrak{l}_{-} = \text{span}\{L_1, L_2, L_3\}$, $\mathfrak{l}_{+} = \mathbb{R} \cdot B$, $\mathfrak{a} \in \{\mathbb{R}^{1,1}, \mathbb{R}^{2,0}\}$ with fixed orthonormal basis A, A_1 , $\alpha = (-\sqrt{2}Z_B \wedge Z_2) \otimes A + (Z_1 \wedge Z_3) \otimes A_1$, $\gamma = tZ_B \wedge Z_1 \wedge Z_3$, $t \in \mathbb{R}$; or
2. $\mathfrak{l} = \mathbb{R} \oplus \mathfrak{h}(1) = \mathbb{R} \cdot L_1 \oplus \{B = [L_2, L_3]_{\mathfrak{l}}\}$, $\mathfrak{l}_{-} = \text{span}\{L_1, L_2, L_3\}$, $\mathfrak{l}_{+} = \mathbb{R} \cdot B$,
 - (a) $\mathfrak{a} \in \{\mathbb{R}^{1,1}, \mathbb{R}^{2,0}\}$ with fixed orthonormal basis A, A_1 , $\alpha = (Z_B \wedge Z_3) \otimes A + (Z_1 \wedge Z_3) \otimes A_1$, $\gamma = (1/\sqrt{2}) \cdot Z_B \wedge Z_1 \wedge Z_2$; or
 - (b) $\mathfrak{a} = \mathbb{R}^{1,0}$, $\alpha = (Z_B \wedge Z_3) \otimes A$, $\gamma = (1/\sqrt{2}) \cdot Z_B \wedge Z_1 \wedge Z_2$;

and

$$\omega = \sqrt{2}(\sigma^{127} + \sigma^{356}) - \sigma^4 \wedge (\sigma^{15} + \sigma^{26} - \sigma^{37}),$$

where $\sigma^1, \dots, \sigma^7$ is the dual basis to $Z_1, Z_2, Z_3, A, L_1, L_2, L_3$.

All listed symmetric triples $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_1, \mathfrak{a})$ are indecomposable and pairwise non-isomorphic.

A direct consequence is the following corollary, which can also be deduced already from Prop. 5.5.

Corollary 6.9 *If an indecomposable symmetric space (M, g) of signature $(4, 3)$ admits a parallel $G_{2(2)}$ -structure, then its transvection group is nilpotent, and its holonomy group is three-dimensional and abelian.*

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